

STAT 302 – Introduction to Probability

Assignment 1 – DUE: 3 October 2025

Problem 1 [10 points]

Suppose we repeatedly roll two fair six-sided dice, and look at the sum of the two values. What is the probability that the first time the sum is exactly 7 is on the 3rd roll?

First solution: We need to look at the outcomes of 3 rolls of 2 fair dice. Consider the sample space consisting of the 6 rolls of the dice, which will have equally likely outcomes:

$$\Omega = \{(a_1, b_1, a_2, b_2, a_3, b_3), a_j, b_j \in \{1, 2, \dots, 6\}\},$$

all outcomes are equally likely, and there are $\#\Omega = 6^6$ of them. The event of interest is

$$A = \{\omega \in \Omega : a_1 + b_1 \neq 7, a_2 + b_2 \neq 7, a_3 + b_3 = 7\}.$$

There are 30 possible values of (a_1, b_1) such that $a_1 + b_1 \neq 7$ (because there are 6 of them that sum up to 7: (1, 6), (2, 5), (3, 4), (4, 3), (5, 2) and (6, 1)). For each of those selections, there are other 30 combinations of a_2 and b_2 that sum other than 7, and finally, for each of those $30 \times 30 = 900$ combinations, there are 6 possibilities for (a_3, b_3) so that they sum to 7. So, $\#A = 30 \times 30 \times 6$, and

$$P(A) = \frac{\#A}{\#\Omega} = \frac{30 \times 30 \times 6}{6^6} = \frac{25}{6^3} = 25/216.$$

Problem 2 [10 points]

An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 7 of them. If a student has figured out how to do 7 of the problems, what is the probability that they

- (a) will answer all 7 problems correctly?
- (b) will answer at least 5 of the problems correctly?

Solution: (a) Of the 10 available problems, there are 7 that the student knows how to solve. The problem is equivalent of asking what is the probability that the 7 problems chosen by the instructor match the 7 that the student knows. The sample space consists of all the possible subsets of 7 problems chosen by the instructor. We assume these are all equally likely. There are "10 choose 7" such choices: $\#\Omega = 120$. Of those subsets, only one contains the 7 problems that the student knows, so the probability of the event of interest is $1/120$.

(b) For the student to answer at least 5 problems correctly, there should be no more than 2 problems that the student does not know. That is: either 0, 1 or 2 problems from the 3 the student does not know. Let A_j be the event that the test contains exactly j problems from those that the student does not know. From part (a), $P(A_0) = 1/120$. For A_1 note that there are "3 choose 1"

choices for the problem the student doesn't know, and for each of those, there are "7 choose 6" ways of selecting the other 6 problems from the ones the student knows, thus,

$$P(A_1) = \frac{\binom{3}{1} \times \binom{7}{6}}{\binom{10}{7}} = \frac{3 \times 7}{120} = \frac{21}{120}.$$

For A_2 note that there are "3 choose 2" choices for the 2 problems the student doesn't know, and for each of those, there are "7 choose 5" ways of selecting the other 5 problems from the ones the student knows, thus,

$$P(A_2) = \frac{\binom{3}{2} \times \binom{7}{5}}{\binom{10}{7}} = \frac{3 \times 21}{120} = \frac{63}{120}.$$

Thus, the answer is

$$P(A_0 \cup A_1 \cup A_2) = \frac{1 + 21 + 63}{120} = \frac{85}{120} = 0.708333.$$

Problem 3 [10 points]

Suppose we roll a fair six-sided die, and flip 6 coins. What is the probability that the number of heads equals the number on the die?

Solution: Consider the sample space $\Omega = \{\omega = (a, x_1, x_2, \dots, x_6), a \in \{1, 2, \dots, 6\}, x_j \in \{0, 1\}\}$, where a denotes the die roll, and $x_j = 1$ if the j -th coin is a head, and $x_j = 0$ otherwise. Then, all $\omega \in \Omega$ are equally likely, and $\#\Omega = 6 \times 2^6 = 6 \times 64 = 384$. For the die roll to be the same as the number of heads in the 6 coin tosses, the outcomes ω should satisfy $a = \sum_{j=1}^6 x_j$. We need to count how many $\omega \in \Omega$ satisfy this restriction. Let

$$A_j = \left\{ \omega = (j, x_1, x_2, \dots, x_6) : \sum_{i=1}^6 x_i = j \right\}.$$

Since $A_j \cap A_\ell = \emptyset$ if $j \neq \ell$, then

$$P(\cup_{j=1}^6 A_j) = \sum_{j=1}^6 P(A_j) = \sum_{j=1}^6 \frac{\#A_j}{\#\Omega} = \frac{\sum_{j=1}^6 \#A_j}{\#\Omega}.$$

Thus, we need to count how many outcomes are in A_j , $j = 1, \dots, 6$. For A_1 , note that for $\sum_{i=1}^6 x_i = 1$ there are only 6 possibilities (you need to select the only i for which $x_i = 1$, all the others have to be zero). For A_2 you need to select 2 of the 6 tosses to have $x_i = 1$ (the other 4 must be zero) so there are "6 choose 2" possible outcomes with $\sum_{i=1}^6 x_i = 2$. For A_3 you can select any 3 of the 6 tosses to have the heads, so there are "6 choose 3" possible outcomes with $\sum_{i=1}^6 x_i = 3$. And similarly for the rest. So

$$P(\cup_{j=1}^6 A_j) = \sum_{j=1}^6 P(A_j) = \sum_{j=1}^6 \binom{6}{j} = \sum_{j=0}^6 \binom{6}{j} - 1 = 2^6 - 1 = 63.$$

Thus, the probability that the number of heads equals the number on the die is $63/384 = 0.1640625$.

Problem 4 [10 points]

There are two urns (# 1 and # 2). Urn # 1 has 5 red and 7 blue balls, while urn # 2 has 6 red and 12 blue balls. We pick three balls at random from each of the two urns (without replacement). What is the probability that all 6 balls are of the same colour?

Solution: Let Ω be the sample space consisting of subsets of size 6 $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, with no particular order, where a_1, a_2 , and a_3 , are the balls selected from the first urn, and b_1, b_2 , and b_3 those from the 2nd urn. There are "12 choose 3" possible selections from the first urn, and for each of those, "18 choose 3" options for the 3 balls drawn from the 2nd urn. So:

$$\#\Omega = \binom{12}{3} \times \binom{18}{3},$$

and all outcomes in Ω are equally likely.

For the 6 balls to be the same colour, they should all be red or all be blue. So, if $A = \{6 \text{ balls of the same colour}\}$, $R = \{6 \text{ red balls}\}$, and $B = \{6 \text{ blue balls}\}$, then $A = R \cup B$, and $R \cap B = \emptyset$. So $P(A) = P(R) + P(B)$.

For R , the 3 balls from each urn have to be red. For the first urn, we have "5 choose 3" such draws. For each of those draws, we have "6 choose 3" draws from the 2nd urn where the 3 balls are red. All these draws are equally likely, so we have

$$P(R) = \frac{\binom{5}{3} \times \binom{6}{3}}{\binom{12}{3} \times \binom{18}{3}} = \frac{10 \times 20}{220 \times 816} = \frac{10}{11 \times 816}$$

Similarly, for B , the 3 balls from each urn have to be blue. For the first urn, we have "7 choose 3" such draws. For each of those draws, we have "12 choose 3" draws from the 2nd urn where the 3 balls are blue. All these draws are equally likely, so we have

$$P(B) = \frac{\binom{7}{3} \times \binom{12}{3}}{\binom{12}{3} \times \binom{18}{3}} = \frac{35}{816}.$$

Finally

$$P(A) = P(R) + P(B) = \frac{35 + 10/11}{816} = 0.04400624$$

Problem 5 [10 points]

Suppose there is a 40% chance that your team wins or ties today's game, and that there is a 30% chance that they win. What is the probability that they tie?

Solution: Let W be the event that my team wins and T be the event that they tie. Thus, we have that $P(W \cup T) = 0.40$ and $P(W) = 0.30$. These events are clearly disjoint, thus $P(W \cup T) = P(W) + P(T)$, so that $0.40 = P(W) + P(T) = 0.30 + P(T)$, and thus $P(T) = 0.10$.

Problem 6 [10 points]

Consider a standard set of poker cards (52 cards, 13 of each of four suits (diamonds, hearts, spades and clubs)). The cards of each suit are numbered 2 to 10, J, Q, K and A. A hand consists of 5 randomly chosen cards. A “full house” consists of a hand with two cards with the same face value, and the other three cards also sharing a face value. For example, (8, 2, 8, 2, 2) is a “full house”, but (Q, 9, Q, Q, Q) is not. If all hands are equally likely, what is the probability of receiving a “full house”?

Solution: We consider the sample space consisting of all possible draws of 5 cards out of the deck, without order. There are “52 choose 5” such draws. To count how many pairs we can draw, select a card face (one of 13), and for each of those, there are 4 cards to choose the pair from, so there are “4 choose 2” such pairs. The total number of possible choices for the first pairs is

$$13 \times \binom{4}{2}$$

For each of these we need to draw the next three cards with the same face value, but different from the face of the pair. Hence, we have 12 face values to choose from, and then “4 choose 3” possible triplets of that face value. The total number of “full house” draws is then

$$13 \times \binom{4}{2} \times 12 \times \binom{4}{3}$$

Since all these draws are equally likely, the probability of obtaining such a draw is

$$\frac{13 \times \binom{4}{2} \times 12 \times \binom{4}{3}}{\binom{52}{5}} = \frac{3744}{2598960} = 0.001440576$$

Problem 7 [10 points]

A set of N people, ($N \geq 3$), including persons A , B and C , are randomly arranged in a line.

- What is the probability that A , B and C sit together? (in other words, that A sits next to B or C , and B sits next to A or C , and C sits next to A or B).
- What would the probability be if the people were randomly arranged in a circle?

Solution: Consider the sample space of all possible arrangements (permutations) of the N people. There $N!$ of these, and all are equally likely. For A , B and C to sit together they need to occupy consecutive seats. They could be the first 3 seats, or the 2nd, 3rd and 4th, or the 3rd, 4th, and 5th, etc. There are $N - 2$ such choices. Once these are chosen, you can re-arrange the 3 friends (in $3!$ ways) and for each of those, still re-arrange the other $N - 3$ people. So, there are

$$3! \times (N - 2) \times (N - 3)! = 3! \times (N - 2)!$$

such possible arrangements. Since they are all equally likely, the probability is

$$\frac{3! \times (N-2) \times (N-3)!}{N!} = \frac{6}{N(N-1)}$$

For part (b), the re-arrangement of the 3 friends, and those of the other $N-3$ people remain the same. But now there are N choices for the sequence of 3 consecutive seats (as long as $N > 3$). Hence we have $3! \times (N-3)! \times N$ possible arrangements, and if $N > 3$ the probability is

$$\frac{3! \times N \times (N-3)!}{N!} = \frac{6}{(N-1)(N-2)}$$

If $N = 3$ then there is only one 3-consecutive seat possibility, and thus the probability is

$$\frac{3! \times (N-3)!}{N!} = \frac{6}{N!} = \frac{6}{3!} = 1$$

(if there are 3 seats in a circle, then A, B and C are always together).

Problem 8 [10 points]

How many different arrangements of the letters in the word “volcanologists” can be obtained?

Solution: The word has 14 letters, so there are $14!$ possible re-arrangements. However, the letter “O” appears 3 times, the letter “S” appears twice, and so does the letter “L”, and repeated letters can be re-arranged resulting in the same letter pattern. So, in the $14!$ permutations each pattern appears $3! \times 2! \times 2! = 24$ times. Thus, the total number of distinct arrangements is $14!/24 = 3,632,428,800$.

Problem 9 [10 points]

For the following claims, state whether they are TRUE or FALSE. Statements claimed to be TRUE must be accompanied by a proof, and statements claimed to be FALSE must be accompanied by a counterexample.

NOTE: For a statement to be TRUE, it **MUST** be TRUE **for any possible** relevant events for the statement in question. For example, for (a) below to be TRUE, it must be true **for any arbitrary events** A and B , not only for independent ones.

- (a) Let A and B be events in the sample space Ω . Then $P(A \cap B) \leq P(A)P(B)$.
- (b) Let E be an event with $0 < P(E) < 1$ and define $P_E(A) = P(A \cap E)$ for every event A in the sample space S . Then P_E satisfies the 3 axioms of probability.
- (c) Let A and B be events in the sample space Ω . Then $P(A \cap B^C) = P(A \cup B) - P(B)$.
- (d) Let $A_1, A_2, \dots, A_n$ be events in the sample space Ω . Then

$$P(\cap_{i=1}^n A_i) \leq \min_i \{P(A_i)\}.$$

Solution: (a) FALSE. Consider an event A with $P(A) = 0.10$, and let $B = A$. Then $P(A \cap B) = P(A) = 0.10$, and $P(A)P(B) = 0.01$, so that $P(A)P(B) < P(A \cap B)$. (b) FALSE. The first axiom would necessitate that $P_E(\Omega) = 1$, however: $P_E(\Omega) = P(\Omega \cap E) = P(E) < 1$. (c) TRUE. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, hence $P(A \cup B) - P(B) = P(A) - P(A \cap B)$. Since, as we saw in class, for any events A and B $P(A) = P(A \cap B) + P(A \cap B^c)$ then $P(A \cup B) - P(B) = P(A) - P(A \cap B) = P(A \cap B^c)$. (d) TRUE. Note that $(\cap_{i=1}^n A_i) \subset A_j$ for any $1 \leq j \leq n$. Thus $P(\cap_{i=1}^n A_i) \leq P(A_j)$ for all $1 \leq j \leq n$, hence the inequality also holds for the smallest of the $P(A_j)$'s: $\min_j P(A_j)$.

Problem 10 [10 points]

Suppose you want to divide a 52 card deck into four hands with 13 cards each.

- (a) What is the probability that each hand has a queen?
- (b) What is the probability exactly two of them have a queen?

Solution: Our sample space consists of all possible permutations of the 52 cards. We will then take the first 13 to be the first hand, the 2nd group of 13 cards to be the 2nd hand, etc. There are $52!$ possible hands.

To count how many of these arrangements have one queen in each hand we will count the possible placements of the 4 queens in the first hand, and the corresponding options for the other cards in the hand. And then count the remaining options for the 2nd hand, and then, having built the 1st and 2nd hand, count options for the 3rd hand, etc.

For the first hand, note that we have 4 choices for the queen (pick any of the 4 available queens). For each of these, there are 13 choices for the spot in the hand where the queen appears. And then you have $48 \times 47 \times \cdots \times 37$ choices for the other 12 cards in that hand. That is

$$4 \times 13 \times 48 \times 47 \times \cdots \times 37$$

possibilities for the first hand. For each of these, in the second hand, you have now 3 queens to choose from, 13 places where to put it, and then $36 \times 35 \times \cdots \times 25$ choices for the other 12 cards. So, for the 1st and 2nd hand combined we have

$$(4 \times 13 \times 48 \times 47 \times \cdots \times 37) \times (3 \times 13 \times 36 \times 35 \times \cdots \times 25)$$

possibilities. For each of those possible arrangements of the first 26 cards, in the 3rd hand you have 2 queens to choose from, 13 places where to put it, and then $24 \times 23 \times \cdots \times 13$ choices for the other 12 cards. So far then, we have

$$4 \times 3 \times 2 \times 13^3 \times 48 \times 47 \times \cdots \times 13$$

possibilities. For each of these possibilities for the first 39 cards, for the last hand we only have one queen left, 13 places where to put it, and $12!$ options for the other 12 cards. Putting it all together, we have

$$4! \times 13^4 \times 48!$$

possible draws, out of a total of $52!$ equally likely possible draws. The probability is

$$\frac{4! \times 13^4 \times 48!}{52!} = 0.1054981993$$

For part (b): There are "4 choose 2" choices for which 2 hands have the queens. For each of those choices, one can have 3 queens in 1 hand, and 1 queen in the other, 2 queens in each, or 1 queen in one hand and 3 in the other. For the (3,1) or (1,3) patterns, we have 13 choices for the place of the 1-queen, for each of those we have "13 choose 3" choices for the places of the 3-queens in other hand, and $4!$ ways of placing the 4 queens in the (3,1) or (1,3) spots. Once these are placed, we can put the other 48 cards in any free slot, there are $48!$ ways of doing that. Thus, the (3,1) patterns appear

$$4! \times 13 \times \binom{13}{3} \times 48!$$

times, and the same is true for the (1,3) patterns. For the (2,2) pattern, we have " 4×3 " options to select the first 2 queens, and 2×1 options for the other 2 queens. For each of these, we can select "13 choose 2" different places to put the queens in one slot, and for each of these another "13 choose 2" choices for the spots of the other 2 queens. And once all this is chosen, we can still put the other 48 cards in any free slot, there are $48!$ ways of doing that. Thus, for the (2,2) pattern we have

$$4! \times \left[\binom{13}{2} \right]^2 \times 48!$$

possibilities. Since all these are equally likely, and recalling that there are "4 choose 2" ways of selecting which 2 hands have the queens, the probability of obtaining such a hand is

$$\begin{aligned} & \frac{\binom{4}{2} \times \left(2 \times [4! \times 13 \times \binom{13}{3} \times 48!] + 4! \times [\binom{13}{2}]^2 \times 48! \right)}{52!} \\ &= \frac{6 \times 48! \times \left(2 \times [4! \times 13 \times \binom{13}{3}] + 4! \times [\binom{13}{2}]^2 \right)}{52!} = 0.2996399 \end{aligned}$$

Problem 8: hilarious ChatGPT answer

Note how hard it works to convince itself it's right, even though itself "can see" (twice) that it's wrong!

Prompt: how many distinct arrangements of the word "volcanologists" can be built?

Answer: We need the number of **distinct permutations** of the letters in **"volcanologists"**.

Step 1: Count letters

Word: **volcanologists**

Length = 13 letters.

Now tally frequencies:

```
* **v**: 1
* **o**: 3
* **l**: 2
* **c**: 1
* **a**: 1
* **n**: 1
* **g**: 1
* **i**: 1
* **s**: 2
* **t**: 1
```

Check total:

$1+3+2+1+1+1+1+1+2+1 = 14$ | too many. Let's recount carefully.

Write it out:

v o l c a n o l o g i s t s

Now count systematically:

```
* v: 1
* o: 3 (positions 2, 7, 9)
* l: 2 (3, 8)
* c: 1
* a: 1
* n: 1
* g: 1
```

```
* i: 1
* s: 2 (12, 14)
* t: 1
```

Total = 1 + 3 + 2 + 1 + 1 + 1 + 1 + 1 + 2 + 1 = **14**.

But the word has 13 letters, so I must have double-counted somewhere. Let's check the spelling:

"volcanologists" has letters:

v (1), o (3), l (2), c (1), a (1), n (1), g (1), i (1), s (1), t (1).

Ah! There is only **one s**, not two.

So corrected frequencies:

```
* v: 1
* o: 3
* l: 2
* c: 1
* a: 1
* n: 1
* g: 1
* i: 1
* s: 1
* t: 1
```

Now total = 13 letters. Good.

Step 2: Formula for distinct arrangements

```
$$
\text{Total} = \frac{13!}{3! \cdot 2!}
$$
```

Step 3: Compute

```
* $13! = 6,227,020,800$
* $3! = 6$, $2! = 2$, so denominator = $12$.
```

```
$$
\frac{13!}{12} = \frac{6,227,020,800}{12} = 518,918,400
```

\$\$

Final Answer:

There are ****518,918,400 distinct arrangements**** of the letters in ***"volcanologists"***.