

STAT 302 – Introduction to Probability
Assignment 2 – DUE: 24 October 2025

SURNAME	First Name
Signature	Student ID

Problem	Points	max.
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1 [10 points]

Assume that the probability of a traffic accident occurring at a certain intersection is 10%. At the same time, the probability that it snows is 20%. Furthermore, suppose that the conditional probability of having an accident, conditional of the event that it snows, is 40%. Compute the conditional probability that it snows given that there is an accident at that intersection.

Solution: Let A be the event that there's a traffic accident at the intersection of interest, and S the event that it snows. Then we know $\mathbb{P}(A) = 0.10$, $\mathbb{P}(S) = 0.20$ and $\mathbb{P}(A|S) = 0.40$. We need $\mathbb{P}(S|A) = \mathbb{P}(A|S) \times \mathbb{P}(S) / \mathbb{P}(A) = 0.40 \times 0.20 / 0.10 = 0.80$.

Problem 2 [10 points]

We roll two fair, 6-sided dice. One of the dice is red, the other one is blue. Let A be the event that the 2 dice show the same face value, B the event that the sum of the two dice is 12, C the event that the red die shows 4, and D that the blue die shows 4.

- (a) Prove whether A and B are independent or not.
- (b) Prove whether A and C are independent or not.
- (c) Prove whether A and D are independent or not.
- (d) Prove whether A , C , and D are independent or not.

Solution: Let the sample space be the two dice rolls, with the red die recorded first, and the blue one recorded second: $\Omega = \{(i, j), 1 \leq i \leq 6, 1 \leq j \leq 6\}$, with all outcomes equally likely, and $\#\Omega = 36$.

- (a) Since $B = \{(6, 6)\}$ and $A = \{(1, 1), (2, 2), \dots, (6, 6)\}$, then $B \subseteq A$ and $A \cap B = B = \{(6, 6)\}$. Thus $\mathbb{P}(A \cap B) = 1/36$. Also $\mathbb{P}(B) = 1/36$ and $\mathbb{P}(A) = \#A / \#\Omega = 6/36 = 1/6$, and hence $1/36 = \mathbb{P}(A \cap B) \neq \mathbb{P}(B) \times \mathbb{P}(A) = (1/36) \times (1/6)$, and A and B are not independent.
- (b) $C = \{(4, j), 1 \leq j \leq 6\}$, then $A \cap C = \{(4, 4)\}$, so $\mathbb{P}(A \cap C) = 1/36$, $\mathbb{P}(A) = 1/6$ and $\mathbb{P}(C) = 6/36 = 1/6$, thus, A and C are independent.
- (c) $D = \{(i, 4), 1 \leq i \leq 6\}$, then $A \cap D = \{(4, 4)\}$, so $\mathbb{P}(A \cap D) = 1/36$, $\mathbb{P}(A) = 1/6$ and $\mathbb{P}(D) = 6/36 = 1/6$, thus, A and D are independent.
- (d) We need to check whether $\mathbb{P}(A \cap C) = \mathbb{P}(A) \times \mathbb{P}(C)$, $\mathbb{P}(A \cap D) = \mathbb{P}(A) \times \mathbb{P}(D)$, $\mathbb{P}(C \cap D) = \mathbb{P}(C) \times \mathbb{P}(D)$, and $\mathbb{P}(A \cap C \cap D) = \mathbb{P}(A) \times \mathbb{P}(C) \times \mathbb{P}(D)$. From (b) and (c) we know that $\mathbb{P}(A \cap C) = \mathbb{P}(A) \times \mathbb{P}(C)$ and $\mathbb{P}(A \cap D) = \mathbb{P}(A) \times \mathbb{P}(D)$. Also, and just as before $C \cap D = \{(4, 4)\}$, thus $\mathbb{P}(C \cap D) = 1/36$ and $\mathbb{P}(C) = \mathbb{P}(D) = 1/6$, and thus $\mathbb{P}(C \cap D) = \mathbb{P}(C) \times \mathbb{P}(D)$. Finally, $A \cap C \cap D = \{(4, 4)\}$, in words, rolling two 4's. Then $\mathbb{P}(A \cap C \cap D) = 1/36$, but $\mathbb{P}(A) \times \mathbb{P}(C) \times \mathbb{P}(D) = (1/6) \times (1/6) \times (1/6) \neq \mathbb{P}(A \cap C \cap D)$, and A , C and D are not independent. Note that "the problem" here, what makes independence fail, is that $C \cap D \subset A$.

Problem 3 [10 points]

We first roll a fair 6-sided die. Next, we toss a number of fair coins equal to the number showing on the die (e.g., if you roll a 5, then you toss 5 coins). Calculate the conditional probability that the number of heads is 3 given that the die roll is a 5.

Solution: Conditional on the die roll being 5, the number of heads is the result of 5 coin tosses, so the sample space consists of all possible such outcomes, equally likely, and there are 32 of them. Of those, there are $\binom{5}{3} = 10$ that contain 3 heads, so the probability of interest is $10/32 = 5/16$.

Note that a much more interesting question (and the one that was actually intended to be in the assignment) would be to calculate the conditional probability that die roll is a 5 conditional on the number of heads being 3. Unfortunately, a mistake was made when typing the question for the assignment....

Problem 4 [10 points]

We have 3 2-sided cards. One is red on both sides, one is blue on both sides, and the third one has one side of each colour. The cards are put in a hat, and one is picked at random and placed on the table (each side is equally likely to be showing). Calculate the conditional probability that the card is the one with 2 red sides, given that the face showing is red.

Solution: The cards can be identified as RR , BB and RB (for the one with 2 red sides, the one with 2 blue sides, and the one with one colour on each side, respectively). We need to calculate the probability that the chosen card is RR conditional on the showing side being red.

We need to calculate $\mathbb{P}(RR|'red') = \mathbb{P}('red'|RR)\mathbb{P}(RR)/\mathbb{P}('red')$. If the chosen card is RR , then the probability of showing a red face is 1. Since the card is selected at random, $\mathbb{P}(RR) = 1/3$. Also: $\mathbb{P}('red') = \mathbb{P}('red'|RR)\mathbb{P}(RR) + \mathbb{P}('red'|BB)\mathbb{P}(BB) + \mathbb{P}('red'|RB)\mathbb{P}(RB) = 1 \times (1/3) + 0 \times (1/3) + (1/2) \times (1/3) = 1/3 + 1/6 = 1/2$. Finally: $\mathbb{P}(RR|'red') = \mathbb{P}('red'|RR)\mathbb{P}(RR)/\mathbb{P}('red') = 1 \times (1/3)/(1/2) = 2/3$.

We can also solve this problem by writing the sample space explicitly. Label the cards as 1, 2, and 3. Then all possible 6 outcomes are $\Omega = \{(1, 'red'), (1, 'red'), (2, 'blue'), (2, 'blue'), (3, 'red'), (3, 'blue')\}$, and they are equally likely. Then $\mathbb{P}(RR|'red') = \mathbb{P}(RR \cap 'red')/\mathbb{P}('red')$. The event $RR \cap 'red' = \{(1, 'red'), (1, 'red')\}$ has probability $2/6 = 1/3$. The event $'red' = \{(1, 'red'), (1, 'red'), (3, 'red')\}$ has probability $3/6 = 1/2$, and thus $\mathbb{P}(RR|'red') = \mathbb{P}(RR \cap 'red')/\mathbb{P}('red') = (1/3)/(1/2) = 2/3$.

Problem 5 [10 points]

Let A and B be two arbitrary events with $P(A \cup B) = 1$. Prove that A and B are independent if and only if $\max\{P(A), P(B)\} = 1$.

Solution: Note that we have $1 = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. If $\max\{P(A), P(B)\} = 1$ then either $\mathbb{P}(A) = 1$ or $\mathbb{P}(B) = 1$. If $\mathbb{P}(A) = 1$, we get $1 = 1 + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, and thus $\mathbb{P}(A \cap B) = \mathbb{P}(B) = \mathbb{P}(B) \times \mathbb{P}(A)$ (because $\mathbb{P}(A) = 1$), and A and B are independent. If $\mathbb{P}(B) = 1$, we get $1 = 1 + \mathbb{P}(A) - \mathbb{P}(A \cap B)$, and thus $\mathbb{P}(A \cap B) = \mathbb{P}(A) = \mathbb{P}(A) \times \mathbb{P}(B)$ (because $\mathbb{P}(B) = 1$), and A and B are independent.

In the other direction, if A and B are independent and $\mathbb{P}(A \cup B) = 1$ we have

$$\begin{aligned} 1 &= \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A) \times \mathbb{P}(B) \\ &= \mathbb{P}(B) + \mathbb{P}(A) \times (1 - \mathbb{P}(B)) \end{aligned}$$

Thus $(1 - \mathbb{P}(B)) = \mathbb{P}(A) \times (1 - \mathbb{P}(B))$. For this to hold, either $1 - \mathbb{P}(B) = 0$ (and thus $\mathbb{P}(B) = 1$), or, if $1 - \mathbb{P}(B) \neq 0$ we get $\mathbb{P}(A) = 1$. Hence, either $\mathbb{P}(B) = 1$ or $\mathbb{P}(A) = 1$, which is $\max\{P(A), P(B)\} = 1$.

Problem 6 [10 points]

We draw 5 cards at random from a standard poker deck with 52 cards (as described in Assignment 1, for example). Label their face values as 1, 2, ..., 13. Let Y be the smallest face value on the 5-card draw. Calculate $E(Y)$, using a computer if needed (in which case you need to include your code in your submitted answer). Note: you should not estimate $E(Y)$, you need to compute its exact value (up to your computer's numerical precision).

Solution: Since there are 4 cards of each face value, the largest value Y can take is 12 (for Y to be 13, we'd need 5 '13's in our hand). So $\mathcal{R}_Y = \{1, 2, \dots, 12\}$. We will first calculate the *CDF*, and we can then either find the PMF, or use the result in the midterm $\mathbb{E}(Y) = \sum_{k=1}^{12} \mathbb{P}(Y \geq k)$.

Our sample space is all possible subsets of size 5 from the 52 cards. There are $\binom{52}{5}$ of them, and they are all equally likely.

We have $\mathbb{P}(Y \leq 12) = 1$, and for $k = 1, 2, \dots, 11$ we have $\mathbb{P}(Y \leq k) = 1 - \mathbb{P}(Y > k) = 1 - \mathbb{P}(Y \geq k+1)$. Now, for $Y \geq k+1$, the 5 drawn cards must have come from those labelled $k+1$ or higher. There are $13 - y$ such labels (in general, there are $13 - b + 1$ labels b or higher), and $4 \times (13 - y)$ such cards. So we have $\binom{4 \times (13 - y)}{5}$ such hands, and

$$\mathbb{P}(Y \leq k) = 1 - \mathbb{P}(Y > k) = 1 - \mathbb{P}(Y \geq k+1) = 1 - \frac{\binom{4 \times (13 - y)}{5}}{\binom{52}{5}}, \quad 1 \leq k \leq 11$$

Here is a simple R script to compute the PMF, and then $\mathbb{E}(Y)$:

```
## the vector of values of F(k), for k \in range of Y
a <- 1 - choose(4*(13-(1:12)), 5) / choose(52, 5)
## f(k) = F(k) - F(k-1)
## first add an entry of 0 for F(0)
a <- c(0, a)
## now a[k] = F(k-1), 1 <= k <= 12
```

```
## we need to calculate
## f(k) = F(k) - F(k-1), 1 <= k <= 12
ff <- a[2:13] - a[1:12]
## E(Y)
sum( ff * (1:12) )
[1] 2.614566
```

Optional checks, not part of the expectations for the assignment We can run a simple sanity check via a short simulation. You are encouraged to try it on your computer:

```
N <- 1e7
ss <- vector('numeric', N)
for(j in 1:N) ss[j] <- min( sample(rep(1:13, each=4), 5) )
# check E(Y)
mean(ss)
# check PMF
cbind(empirical=round(table(ss)/N, 5), theoretical=round(ff, 5))
```

Alternatively, we can also compute $\mathbb{E}(Y)$ from the CDF using the result in the midterm. In our case we have

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{k=1}^{+\infty} \mathbb{P}(Y \geq k) = \sum_{k=1}^{12} \mathbb{P}(Y \geq k) \\ &= \sum_{k=1}^{12} 1 - \mathbb{P}(Y < k) = \sum_{k=1}^{12} 1 - \mathbb{P}(Y \leq k-1) \\ &= \sum_{k=1}^{12} 1 - F_Y(k-1)\end{aligned}$$

Recall that in our script above the k -th element of the vector a contains $F_Y(k-1)$. Hence, to compute $\mathbb{E}(Y)$ we could use

```
## E(Y) via the CDF
sum( (1 - a[1:12]) )
[1] 2.614566
```

Problem 7 [10 points]

We toss 2 coins. Let $X = 1$ if the first coin is "heads", and $X = 0$ otherwise. Let $Y = 1$ if the second coin is "heads", and $Y = 5$ otherwise. Let $Z = XY$. Find the CDF of Z .

Solution: The sample space is the set of 4 equally likely outcomes of the 2 tosses: (H, H) , (H, T) , (T, H) , and (T, T) . We have $\mathcal{R}_Z = \{1, 0, 5\}$. Thus: $f_Z(1) = \mathbb{P}(Z = 1) = \mathbb{P}(X = 1, Y = 1) = \mathbb{P}((H, H)) = 1/4$; $f_Z(0) = \mathbb{P}(Z = 0) = \mathbb{P}(X = 0) = \mathbb{P}(\{(T, H), (T, T)\}) = 1/4 + 1/4 = 1/2$; and $f_Z(5) = \mathbb{P}(Z = 5) = \mathbb{P}(X = 1, Y = 5) = \mathbb{P}((H, T)) = 1/4$. Hence:

$$F_Z(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1/2 & \text{if } 0 \leq a < 1 \\ 3/4 & \text{if } 1 \leq a < 5 \\ 1 & \text{if } 5 \leq a \end{cases}$$

Problem 8 [10 points]

There is a loop in a computer program, which exits with probability $1/3$, every time it runs. What is the probability that the loop runs 5 times before exiting?

Solution: If A_j is the event that the loop exits in the j -th run, then $\mathbb{P}(A_j) = 1/3$. Implicit in the question is the assumption that different runs of the loop are independent of each other, in other words, that the events A_j , are independent of each other. We need to compute $\mathbb{P}(A_1^c \cap A_2^c \cdots A_5^c \cap A_6)$. Note that $\mathbb{P}(A_1^c \cap A_2^c \cdots A_5^c \cap A_6) = \mathbb{P}(A_1^c) \times \mathbb{P}(A_2^c) \cdots \mathbb{P}(A_5^c) \times \mathbb{P}(A_6)$, because if $A_1, \dots, A_6$ are independent, then all combinations of them and their complements are also independent, as we discussed in class. Hence, $\mathbb{P}(A_1^c \cap A_2^c \cdots A_5^c \cap A_6) = (2/3)^5 \times (1/3) = 0.04389575$.

Problem 9 [10 points]

Let X be a random variable with $\mathcal{R}_X = \mathbb{N} = \{1, 2, \dots\}$. Its PMF is $f_X(a) = 2^{-a}$ for $a \in \mathcal{R}_X$, and $f_X(a) = 0$ if $a \notin \mathcal{R}_X$. Calculate $V(X)$. Hint: the distribution of $Z = X - 1$ belongs to one of the families discussed in class.

Solution: We have $f_X(a) = (1/2)^a = (1/2) \times (1/2)^{a-1}$ for $a = 1, 2, \dots$, and recognize that $X \sim \text{Geom}(1/2)$. We have seen in class that if $Y \sim \text{Geom}(p)$ then $V(Y) = (1-p)/p^2$. Thus, we have $V(X) = (1 - 1/2)/(1/2)^2 = (1/2)/(1/4) = 2$.

Problem 10 [10 points]

Let X be the value obtained by the roll of a fair 6-sided die, and let $Y = |X - 3|$.

(a) Find the CDF of Y .

(b) Find $E(Y)$ and $V(Y)$.

Solution: We have $\mathcal{R}_Y = \{0, 1, 2, 3\}$

- (a) We have $f_Y(0) = \mathbb{P}(Y = 0) = \mathbb{P}(X = 3) = 1/6$; $f_Y(1) = \mathbb{P}(Y = 1) = \mathbb{P}(\{X = 4\} \cup \{X = 2\}) = 1/6 + 1/6 = 1/3$ (because the two events are disjoint); $f_Y(2) = \mathbb{P}(Y = 2) = \mathbb{P}(\{X = 1\} \cup \{X = 5\}) = 1/3$ (because the two events are disjoint); and finally, $f_Y(3) = \mathbb{P}(Y = 3) = \mathbb{P}(\{X = 6\}) = 1/6$. Thus

$$F_Y(a) = \begin{cases} 0 & \text{if } a < 0 \\ 1/6 & \text{if } 0 \leq a < 1 \\ 1/2 & \text{if } 1 \leq a < 2 \\ 5/6 & \text{if } 2 \leq a < 3 \\ 1 & \text{if } 3 \leq a \end{cases}$$

- (b) $\mathbb{E}(Y) = 0 \times (1/6) + 1 \times (1/3) + 2 \times (1/3) + 3 \times (1/6) = 1/3 + 2/3 + 1/2 = 3/2$. Also, $\mathbb{E}(Y^2) = 0^2 \times (1/6) + 1^2 \times (1/3) + 2^2 \times (1/3) + 3^2 \times (1/6) = 1/3 + 4/3 + 9/6 = 19/6$. Thus $V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 19/6 - (3/2)^2 = 19/6 - 9/4 = (38 - 27)/12 = 11/12$.