STAT 302 – Introduction to Probability

Assignment 3 – DUE: 14 November 2025

SURNAME	First Name
Signature	Student ID

Problem	Points	max.
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1 [10 points]

On a specific day, a car salesperson has scheduled 2 sales appointments. His first appointments lead to a sale with probability 0.3, and second ones lead independently to a sale with probability 0.6. Any sale made is equally likely to be either for a high-end car with a $120,000 sale price, or an economy car which costs $35,000. Let X be the dollar amount of the salesperson total sales for this day. Compute $\mathbb{E}(X)$.

Solution:

First, determine $\mathcal{R}_X = \{\$0, \$35K, \$70K, \$120K, \$155K, \$240K\}$ (to simplify the notation, and since there is no ambiguity, we will ignore the "$" sign, and the K units, from now on). In other words, we can work with $Y = X/1000$ and then $\mathcal{R}_Y = \{0, 35, 70, 120, 155, 240\}$. Clearly, $\mathbb{E}[X] = \mathbb{E}[1000 \times Y] = 1000 \times \mathbb{E}[Y]$.

Let $S_1 = 1$ if a sale occurs in their first appointment ($S_1 = 0$ otherwise), and let S_2 be another random variable, independent from S_1, such that $S_2 = 1$ if a sale occurs in the second appointment ($S_2 = 0$ otherwise). Let C be a random variable such that $\mathbb{P}(C = 35) = \mathbb{P}(C = 120) = 1/2$, which indicates what car is sold. Now $Y = 0$ if and only if $S_1 = S_2 = 0$. This event has $\mathbb{P}(Y = 0) = \mathbb{P}(S_1 = 0, S_2 = 0) = \mathbb{P}(S_1 = 0) \times \mathbb{P}(S_2 = 0) = (1 - 0.3) \times (1 - 0.6) = 0.28$ (because of the independence between S_1 and S_2). Next, $\{Y = 35\} = \{S_1 = 1, S_0 = 0, C = 35\} \cup \{S_1 = 0, S_0 = 1, C = 35\}$, and these two events are disjoint. We have

$$
\begin{aligned}
&\mathbb{P}(Y = 35) = \mathbb{P}(S_1 = 1, S_0 = 0, C = 35) + \mathbb{P}(S_1 = 0, S_0 = 1, C = 35) \\
&= \mathbb{P}(C = 35 | S_1 = 1, S_0 = 0)\mathbb{P}(S_1 = 1, S_0 = 0) + \mathbb{P}(C = 35 | S_1 = 0, S_0 = 1)\mathbb{P}(S_1 = 0, S_0 = 1) \\
&\qquad = (1/2) \times 0.3 \times 0.4 + (1/2) \times 0.7 \times 0.6 = (1/2)0.12 + (1/2)0.42 = 0.27
\end{aligned}
$$

Similarly, $\mathbb{P}(Y = 120) = 0.27$, $\mathbb{P}(Y = 70) = \mathbb{P}(Y = 240) = 0.045$, and $\mathbb{P}(Y = 155) = 0.09$. As a quick sanity check, note that $0.28 + 0.27 + 0.27 + 0.045 + 0.045 + 0.09 = 1$. Simple arithmetic then gives

$$
\mathbb{E}[Y] = 0 \times 0.28 + 35 \times 0.27 + 70 \times 0.045 + 120 \times 0.27 + 155 \times 0.09 + 240 \times 0.045 = 69.75
$$

And thus: $\mathbb{E}[X] = \$1000 \times \mathbb{E}[Y] = \$69750$.

Problem 2 [10 points]

Suppose there are 2 urns. Urn number 1 contains 100 chips: 30 are labelled 1's, 40 are labelled 2's, and 30 are labelled 3's. Urn number 2 contains 100 chips: 20 are labelled 1's, 50 are labelled 2's, and 30 are labelled 3's. A coin is tossed and if a head is observed then a chip is randomly drawn from Urn 1, otherwise a chip is drawn from Urn 2. Let Y be the value of the drawn chip. If the occurrence of a head on the coin is denoted by $X = 1$, a tail by $X = 0$, and $X \sim Bi(1, 3/4)$, find:

(a) $\mathbb{E}(X|Y)$

(b) $\mathbb{E}(Y|X)$

(c) $\mathbb{E}(Y)$

(d) $\mathbb{E}(X)$

Solution:

(a) We need to find $\mathbb{P}(X = 1|Y = k) = 1 - \mathbb{P}(X = 0|Y = k)$ for $k = 1, 2, 3$. Since we can compute $\mathbb{P}(Y = k|X = 1)$ and $\mathbb{P}(Y = k|X = 0)$ (because X indicates which urn is used), then we will use that

$$\mathbb{P}(X = 1|Y = k) = \mathbb{P}(Y = k|X = 1)\mathbb{P}(X = 1)/\mathbb{P}(Y = k)$$

The two terms in the denominator are known, for $\mathbb{P}(Y = k)$ we have

$$f_Y(k) = \mathbb{P}(Y = k) = \mathbb{P}(Y = k|X = 1)\mathbb{P}(X = 1) + \mathbb{P}(Y = k|X = 0)\mathbb{P}(X = 0)$$

Thus, $f_Y(1) = (30/100)(3/4)+(20/100)(1/4) = 11/40$, $f_Y(2) = (40/100)(3/4)+(40/100)(1/4) = 17/40$, and $f_Y(3) = (30/100)(3/4) + (30/100)(1/4) = 12/40$. Hence

$$\mathbb{P}(X = 1|Y = 1) = \mathbb{P}(Y = 1|X = 1)\mathbb{P}(X = 1)/\mathbb{P}(Y = 1) = (30/100)(3/4)/(11/40) = 9/11$$

and

$$\mathbb{P}(X = 0|Y = 1) = \mathbb{P}(Y = 1|X = 0)\mathbb{P}(X = 0)/\mathbb{P}(Y = 1) = (20/100)(1/4)/(11/40) = 2/11$$

so that $\mathbb{E}(X|Y = 1) = 0 \times 2/11 + 1 \times 9/11 = 9/11$. Similar calculations give

$$\mathbb{P}(X = 1|Y = 2) = 12/17\,, \qquad \mathbb{P}(X = 0|Y = 2) = 5/17$$

and

$$\mathbb{P}(X = 1|Y = 3) = 3/4\,, \qquad \mathbb{P}(X = 0|Y = 3) = 1/4$$

Hence $\mathbb{E}(X|Y = 2) = 12/17$ and $\mathbb{E}(X|Y = 3) = 3/4$.

For (b), we have $f_{Y|X=1}$ and $f_{Y|X=0}$, so that $E(Y|X = 1) = 1\times(30/100)+2\times(40/100)+3\times(30/100) = 30/100 + 80/100 + 90/100 = 2$ and $E(Y|X = 0) = 1(2/10) + 2(5/10) + 3(3/10) = 21/10$.

For (c), since we computed f_Y in part (a), we can calculate $\mathbb{E}[Y] = 1 \times 11/40 + 2 \times 17/40 + 3 \times 12/40 = 81/40$. As a quick sanity check, we can verify that $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y|X = 1]\mathbb{P}(X = 1) + \mathbb{E}[Y|X = 0]\mathbb{P}(X = 0) = 2 \times 3/4 + (21/10)(1/4) = 60/40 + 21/40 = 81/40 = \mathbb{E}[Y]$.

For (d), $\mathbb{E}[X] = 3/4$, because $X \sim Bi(1, 3/4)$, or simply $E[X] = \mathbb{P}(X = 1) = 3/4$. As a quick sanity check, we can verify that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X|Y = 1]\mathbb{P}(Y = 1) + \mathbb{E}[X|Y = 2]\mathbb{P}(Y = 2) + \mathbb{E}[X|Y = 3]\mathbb{P}(Y = 3) = (9/11)(11/40) + (12/17)(17/40) + (3/4)(12/40) = 9/40 + 12/40 + 9/40 = 3/4 = \mathbb{E}[X]$.

Problem 3 [10 points]

The monthly worldwide average number of nuclear plant critical incidents is 3.5. Let X be the number of upcoming consecutive incident-free quarters, starting in 2026. There are 4 quarters in a year. In other words, X is the number of incident-free quarters observed in a row, starting January 1, 2026, say. For example, if the next incident occurs after 10 months, then $X = 3$. If $X = 2$, there were no incidents in the first 6 months, and at least one incident between months 6 and 9. What is $\mathbb{E}(X)$?

Solution:

Let Y be the number of events in a quarter (a 3-month period), then $Y \sim \mathcal{P}(10.5)$. Since the number of events in disjoint periods of time are independent random variables, then we can think of the binomial experiment of observing whether a quarter is incident-free or not. Each of those "experiments" have constant probability of "success" $p = \mathbb{P}(Y = 0) = e^{-10.5}$, and then if X is the number of incident-free quarters in a row, then $X = W - 1$, where W counts how many "experiments" we need to run until we see our first incident, then $W \sim Geom(1 - p)$ (a success is having an incident in a quarter, which has probability $q = \mathbb{P}(Y > 0) = 1 - \mathbb{P}(Y = 0) = 1 - e^{-10.5}$) Then $\mathbb{E}[W] = 1/q = 1/(1-e^{-10.5}) = e^{10.5}/(e^{10.5}-1)$ and $\mathbb{E}[X] = \mathbb{E}[W]-1 = e^{10.5}/(e^{10.5}-1)-1 = 1/(e^{10.5} - 1) = 2.753721e - 05$, almost zero.

Problem 4 [10 points]

Consider a box containing the 13 diamond cards from a regular deck of cards. Assume, for simplicity, that they are numbered 1, 2, . . . , 13. We randomly draw 3 cards without replacement. Let X be the lowest card face (number) in the draw, and Y be the highest card face in the draw. Prove whether X and Y are independent.

Solution:

It would be sufficient to find a pair k_1 and k_2 with $\mathbb{P}(X = k_1, Y = k_2) \neq \mathbb{P}(X = k_1) \times \mathbb{P}(Y = k_2)$. Note that when $X = Y$ then, all 3 cards must have the same face value. The sample space is all possible triples of cards, with a total of $52 \times 51 \times 50$ equally likely outcomes. Then, for example $\mathbb{P}(X = 1, Y = 1)$ is the probability that the 3 drawn cards have the face value 1, there are $4 \times 3 \times 2$ such possible outcomes. Furthemore, note that if $\{Y = 1\}$, then all cards must be ones, and in particular, $\{X = 1\}$ as well, thus $\{Y = 1\} \subseteq \{X = 1\}$, and then $\{X = 1\} \cap \{Y = 1\} = \{Y = 1\}$, so that $\mathbb{P}(X = 1, Y = 1) = \mathbb{P}(Y = 1)$. Since clearly $\mathbb{P}(X = 1) < 1$ (because there are outcomes where the minimum face value is higher than 1), then $\mathbb{P}(X = 1, Y = 1) = \mathbb{P}(Y = 1) > \mathbb{P}(Y = 1) \times \mathbb{P}(X = 1)$ and thus X and Y are not independent.

Problem 5 [10 points]

Let $X \sim \mathcal{P}(\lambda)$ for some $\lambda > 0$. We have $Y|X = k \sim Bi(k, p)$, where $p \in (0, 1)$. Prove that $X|Y = 0 \ \sim \ \mathcal{P}\left(\lambda(1-p)\right)$.

Solution:

Since $\mathcal{R}_X = \mathbb{N}$, let's try to calculate $\mathbb{P}(X = k|Y = 0)$ for $k \in \mathbb{N}$. We have, for any $k \in \mathbb{N}$:

$$\mathbb{P}(X = k|Y = 0) = \mathbb{P}(Y = 0|X = k) \times \mathbb{P}(X = k)/\mathbb{P}(Y = 0)$$

$$= \frac{\binom{k}{0}p^0(1-p)^k \times e^{-\lambda}\lambda^k/k!}{e^{-\lambda p}}$$

$$= \frac{\left[(1-p)\lambda\right]^k \ e^{\lambda p - \lambda}}{k!} = e^{-\lambda(1-p)} \ \frac{\left[(1-p)\lambda\right]^k}{k!}$$

where we have used that under the conditions of the problem, $\mathbb{P}(Y = 0) = e^{-\lambda p}$. To show this, simply calculate it:

$$\mathbb{P}(Y = 0) = \sum_{k=0}^{\infty} \mathbb{P}(Y = 0, X = k) = \sum_{k=0}^{\infty} \mathbb{P}(Y = 0|X = k)\mathbb{P}(X = k)$$

$$= \sum_{k=0}^{\infty} \binom{k}{0} p^0(1-p)^k e^{-\lambda}\lambda^k/k!$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \left[(1-p)\lambda\right]^k /k! = e^{-\lambda} \, e^{(1-p)\lambda} = e^{-\lambda p}$$

Finally, note that the PMF above for $X|Y = 0$ is that of a $\mathcal{P}\left(\left[(1-p)\lambda\right]\right)$ random variable.

Problem 6 [10 points]

Find:

(a) $\mathbb{E}[(X-2)^2]$, if $\mathbb{E}(X) = 0$ and Var(X) $= 2$.

(b) Var$(5X^2)$, if $\mathcal{R}_X = \{-1, 0, 1\}$, $\mathbb{E}(X) = 0$, and Var(X) $= 1/2$.

(c) $V(a^Y)$, if $a > 0$ and $Y \sim \mathcal{P}(\lambda)$, for some $\lambda > 0$.

Solution:

For (a): From the MT, $\mathbb{E}[(X-2)^2] = (\mathbb{E}[X]-2)^2 + V(X) = (0-2)^2 + 2 = 6$.
For (b): $\mathsf{Var}(5X^2) = 25\mathsf{Var}(X^2) = 25\,[E[(X^2)^2] - (E[X^2])^2] = 25\,[E[X^4] - (E[X^2])^2]$. Note that $E[X^4] = (-1)^4\,\mathbb{P}(X=-1) + 0^4\,\mathbb{P}(X=0) + 1^4\,\mathbb{P}(X=1) = (-1)^2\,\mathbb{P}(X=-1) + 0^2\,\mathbb{P}(X=0) + 1^2 = \mathbb{E}[X^2] = \mathsf{Var}(X) + (\mathbb{E}[X])^2 = 1/2$. Also $(\mathbb{E}[X^2])^2 = (1/2)^2 = 1/4$. Thus $\mathsf{Var}(5X^2) = 25\,[E[X^4] - (E[X^2])^2] = 25\,[\mathbb{E}[X^2] - (E[X^2])^2] = 25[1/2 - 1/4] = 25/4$.
For (c), $\mathsf{Var}(a^Y) = \mathbb{E}((a^Y)^2) - (\mathbb{E}[a^Y])^2 = \mathbb{E}((a^2)^Y) - (\mathbb{E}[a^Y])^2$. For any $b > 0$ we have

$$\mathbb{E}[b^Y] = \sum_{k=1}^{\infty} b^k \mathbb{P}(Y=k) = \sum_{k=1}^{\infty} b^k e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{(\lambda b)^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{(\lambda b)^k}{k!}$$

$$= e^{-\lambda}\, e^{\lambda b} = e^{\lambda(b-1)}$$

Thus, $\mathbb{E}[a^Y] = e^{\lambda(a-1)}$ and $\mathbb{E}((a^Y)^2) = \mathbb{E}((a^2)^Y)] = e^{\lambda(a^2-1)}$. Finally

$$\mathsf{Var}(a^Y) = \mathbb{E}((a^Y)^2) - (\mathbb{E}[a^Y])^2 = e^{\lambda(a^2-1)} - e^{2\,\lambda(a-1)}$$

Problem 7 [10 points]

We roll a fair 6-sided die. Let X be its observed value. We then we toss X fair coins, and let Y be the number of heads obtained. Using a computer if necessary, find $\mathbb{E}(Y)$ and $\mathbb{E}(X|Y=0)$.

Solution:

First note that $\mathcal{R}_Y = \{0,1,2,\ldots,6\}$ (because $\mathcal{R}_X = \{1,2,\ldots,6\}$, and Y is the number of heads obtained from X coin tosses). To compute $\mathbb{E}[Y]$, we can find the pdf of Y by calculating $\mathbb{P}(Y=k)$ for $k = 0,\ldots,6$. By the "Total Probability Theorem", we have

$$\mathbb{P}(Y=k) = \sum_{n=1}^{6} \mathbb{P}(Y=k|X=n)\mathbb{P}(X=n)\,, \qquad k = 0,\ldots,6$$

Note that if $Y=k$, then, necessarily $X \geq k$. Alternatively, for all $\ell < k$ we have $\{X \leq \ell\} \subset \{Y < k\} \subset \{Y \neq k\}$ so that $\{X < \ell\} \cap \{Y = k\} = \varnothing$ for $\ell \leq k-1$, and hence

$$\mathbb{P}(Y=k|X=n) = 0 \qquad \text{for } n < k$$

We have, if $1 \le k \le 6$:

$$\begin{aligned}\mathbb{P}(Y = k) &= \sum_{n=1}^{6} \mathbb{P}(Y = k|X = n)\mathbb{P}(X = n) = \sum_{n=k}^{6} \mathbb{P}(Y = k|X = n)\mathbb{P}(X = n) \\ &= \sum_{n=k}^{6} \binom{n}{k} (1/2)^n (1/6) = (1/6) \sum_{n=k}^{6} \binom{n}{k} (1/2)^n\end{aligned}$$

For $k = 0$ we have

$$\mathbb{P}(Y = 0) = \sum_{n=1}^{6} \mathbb{P}(Y = 0|X = n)\mathbb{P}(X = n) = \sum_{n=1}^{6} (1/2)^n (1/6) = (1/6)(1 - (1/2)^6)$$

Using the computer we get, for $k = 0$:

```
pp <- 1/2
fy <- vector('numeric', 7)
fy[1] <- 1/6 * sum( pp^(1:6) ) # same as "(1/6)*(1 - (1/2)^6)"
fy[1]
[1] 0.1640625
```

For $k = 1, \ldots, 6$ we get

```
for(k in 2:7)
  fy[k] <- (1/6)*sum( dbinom(k-1, size=(k-1):6, prob=pp) )
> round(fy, 7)
[1] 0.1640625 0.3125000 0.2578125 0.1666667 0.0755208 0.0208333 0.0026042
```

Finally, $\mathbb{E}[Y] = \sum_{k=0}^{6} k\, f_Y(k)$ which is

```
> sum( fy * 0:6 )
[1] 1.75
```

To calculate $\mathbb{E}[X|Y = 0]$ we calculate $\mathbb{P}(X = k|Y = 0)$ for $k = 1, \ldots, 6$:

$$\begin{aligned}\mathbb{P}(X = k|Y = 0) &= \mathbb{P}(Y = 0|X = k)\mathbb{P}(X = k)/\mathbb{P}(Y = 0) \\ &= (1/2)^k (1/6) / \left[(1/6)(1 - (1/2)^6\right] = (1/2)^k / \left[1 - (1/2)^6\right]\end{aligned}$$

Recall that $1 - (1/2)^6 = \sum_{j=1}^{6} (1/2)^j$ and using the computer we get

```
> pp2 <- pp^(1:6) / sum( pp^(1:6) ) # P(X = k | Y =0)
> sum(pp2*(1:6)) # sum_{k=1}^6 k * P(X = k | Y =0)
[1] 1.904762
```

Problem 8 [10 points]

Let X be a discrete random variable with $f_X(-1) = f_X(1) = 1/4$ and $f_X(0) = 1/2$. Let $Y = X^2$. Show that $cov(X, Y) = 0$ but X and Y are not independent.

Solution:

We have $cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XX^2] - \mathbb{E}[X]\mathbb{E}[X^2] = \mathbb{E}[X^3]$ (because $\mathbb{E}[X] = 0$). Also $\mathbb{E}[X^3] = (-1)^3 \times \mathbb{P}(X = -1) + 0 \times \mathbb{P}(X = 0) + 1^3 \times \mathbb{P}(X = 1) = -f_X(-1) + f_X(1) = 0$, thus $cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[XX^2] - \mathbb{E}[X]\mathbb{E}[X^2] = \mathbb{E}[X^3] = 0$.

Since $Y = X^2$, then $\mathbb{P}(X = 1, Y = 1) = \mathbb{P}(X = 1) = 1/4$ (because $\{X = 1\} \subseteq \{Y = 1\}$). And also $\mathbb{P}(Y = 1) = \mathbb{P}(\{X = -1\} \cup \{X = 1\}) = 1/4 + 1/4 = 1/2$. Thus $\mathbb{P}(X = 1, Y = 1) = 1/4$ and $\mathbb{P}(X = 1)\mathbb{P}(Y = 1) = (1/4)(1/2) = 1/8$, thus X and Y are not independent.

Problem 9 [10 points]

Consider a discrete random vector (X_1, X_2) with point mass probability function given by

		X_2		
		0	1	2
X_1	-1	$1/6$	0	$1/6$
	0	0	a	a
	1	$1/6$	$1/6$	0

for some $a \in [0,1]$. Note that $\mathcal{R}_{X_1} = \{-1, 0, 1\}$ and $\mathcal{R}_{X_2} = \{0, 1, 2\}$.

(a) If $V(X_2) = 2/3$, compute $V(X_1 - 2\,X_2)$.

(b) Compute $E(X_1|X_2)$.

Solution:

We can find the value of a using that the PMF should satisfy: $\sum_{k_1}\sum_{k_2} f_{(X_1,X_2)}(k_1,k_2) = 1$. Thus $1/6 + 1/6 + 2a + 1/6 + 1/6 = 1$, and $2/3 + 2a = 1$, and $a = 1/6$.

For (a) note that $V(X_1 - 2X_2) = V(X_1) + V(2\,X_2) - 2cov(X_1, 2\,X_2) = V(X_1) + 4\,V(X_2) - 4\,cov(X_1, X_2)$. To compute $V(X_1)$ we need the PMF of X_1 which is $f_{X_1}(-1) = \sum_{k_2} f_{(X_1,X_2)}(-1,k_2) = 1/6 + 1/6 = 1/3$, and similarly for $f_{X_1}(0) = f_{X_1}(1) = 1/3$. Then $\mathbb{E}[X_1] = -1(1/3) + 0(1/3) + 1(1/3) = 0$, and $\mathbb{E}[X_1^2] = (-1)^2(1/3) + 0^2(1/3) + 1^2(1/3) = 2/3$, thus $V(X_1) = 2/3 - 0 = 2/3$. To calculate $cov(X_1, X_2) = \mathbb{E}[X_1\,X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \mathbb{E}[X_1\,X_2] - 0\,\mathbb{E}[X_2] = \mathbb{E}[X_1\,X_2]$. Simple calculations give $\mathbb{E}[X_1\,X_2] = (-2)(1/6) + 1(1/6) = 1/6 - 1/3 = -1/6$. Finally: $V(X_1 - 2X_2) = V(X_1) + 4\,V(X_2) - 4\,cov(X_1, X_2) = 2/3 + 4\,(2/3) - 4(-1/6) = 2/3 + 8/3 + 2/3 = 4$.

For (b) we need to compute $E(X_1|X_2 = k_2)$ for different values of $k_2 \in \{0,1,2\}$. The conditional PMF of $X_1|X_2 = k_2$ satisfies $f_{X_1|X_2=k_2}(k_1) = f_{(X_1,X_2)}(k_1,k_2)/f_{X_2}(k_2)$, so we need to find the marginal PMF of X_2. We have $f_{X_2}(0) = f_{X_2}(1) = f_{X_2}(2) = 1/3$.

When $X_2 = 0$ we have $f_{X_1|X_2=0}(-1) = f_{X_1|X_2=1}(1) = (1/6)/(1/3) = 1/2$ and $f_{X_1|X_2=0}(0) = 0$. So that $\mathbb{E}[X_1|X_2 = 0] = (-1)(1/2) + 1(1/2) = 0$.

When $X_2 = 1$ we have $f_{X_1|X_2=1}(-1) = 0$ and $f_{X_1|X_2=1}(0) = f_{X_1|X_2=1}(1) = (1/6)/(1/3) = 1/2$, and $\mathbb{E}[X_1|X_2 = 1] = (0)(1/2) + 1(1/2) = 1/2$.

When $X_2 = 2$ we have $f_{X_1|X_2=2}(1) = 0$ and $f_{X_1|X_2=2}(-1) = f_{X_1|X_2=1}(0) = (1/6)/(1/3) = 1/2$, and $\mathbb{E}[X_1|X_2 = 2] = (-1)(1/2) + 0(1/2) = -1/2$.

As a sanity check note that $\mathbb{E}[\mathbb{E}[X_1|X_2]] = 0(1/3) + (1/2)(1/3) + (-1/2)(1/3) = 0 = \mathbb{E}[X_1]$.

Problem 10 [10 points]

Let X and Y be two random variables, not necessarily independent, such that $X \sim Bi(1, p)$ and $Y \sim Bi(1, q)$, where $p, q \in (0, 1)$. Let $Z = X + Y$ and $W = X - Y$. Show that if $p + q = 1$ then $cov(Z, W) = 0$, and check whether Z and W are also independent or not.

Note: Such non-independent X and Y can be constructed as follows. Let $X \sim Bi(1, p)$, with $p < 1/2$, and let Y be a function of X as follows: if $X = 1$, then $Y = 1$; if $X = 0$ then we toss a biased coin with $\mathbb{P}(Heads) = \alpha$, and let $Y = 1$ if you observe heads, and $Y = 0$ otherwise. Then Y has a Binomial distribution, and if we choose $\alpha = (1-2p)/(1-p)$, then $\mathbb{P}(X = 1)+\mathbb{P}(Y = 1) = 1$, and X and Y are not independent.

Solution:

We have $cov(Z, W) = cov(X + Y, X - Y) = cov(X + Y, X) - cov(X + Y, Y) = cov(X, X) + cov(Y, X) - cov(X, Y) - cov(Y, Y) = V(X) - V(Y)$. Since $X \sim Bi(1, p)$ and $Y \sim Bi(1, q)$, we have $V(X) = 1 \times p(1 - p)$ and $V(Y) = 1 \times q(1 - q)$. If $p + q = 1$, then $q = 1 - p$ and $p = 1 - q$, thus $V(X) = V(Y)$ and $cov(Z, W) = V(X) - V(Y) = 0$.

Z and W may be independent or not. For example, with the construction in the note above, they are not independent. But if $p = 1/2$, and $Y = 1 - X$, then Z and W are independent.