STAT 302 – Introduction to Probability

Assignment 4 – DUE: 28 November 2025

SURNAME	First Name
Signature	Student ID

Problem	Points	max.
1		10
2		10
3		10
4		10
5		10
~~6~~		~~10~~
7		10
8		10
9		10
10		10
Total:		~~100~~ 90

Problem 1

The time it takes to connect a caller with a customer service representative (in hours) is an exponential random variable with expected value λ.

(a) [3 points] If $\lambda = 3$, what is the probability that the next time you call you will need to wait less than one hour to be connected with a person ?

(b) [7 points] The number of people calling the call centre in any one hour period follows a $\mathcal{P}(10)$ distribution. What should $\lambda > 0$ be so that the probability that none of the customers calling in the next hour has to wait more than hour to be connected, is at least 95%?

Solution: The statement of the problem may be misinterpreted. If T denotes the time it takes to connect a caller with a customer service representative (in hours), and T has an $\mathcal{E}(\alpha)$ distribution, then, following the notation used in class, we have $\mathbb{E}[T] = 1/\alpha$. Thus, if, as the problem says, λ is the expected value of T, then $\lambda = 3$ would mean that $T \sim \mathcal{E}(1/3)$. However, in class, we always used the letter λ to denote the **parameter** of the Exponential distribution, **not its expected value**. Solutions with either of the two interpretations will be accepted.

1. If you took λ to denote **the parameter** of the distribution of T (i.e. if $T \sim \mathcal{E}(\lambda)$):

   (a) Let T be the time it takes to connect a caller with a customer service representative (in hours). Then $T \sim \mathcal{E}(3)$. Waiting less than one hour is the event $\{T < 1\}$ which has probability $\mathbb{P}(T < 1) = F_T(1) = 1 - e^{-3\times 1} = 1 - e^{-3} \approx 0.95$.

   (b) Let N be the number of people calling the call centre in any one hour period, then $N \sim \mathcal{P}(10)$. Let Y be the number of customers calling in the next hour that have to wait more than hour to be connected. Then, if $T \sim \mathcal{E}(\lambda)$, then $Y|N = n \sim Bi(n, q)$ with $q = \mathbb{P}(T > 1) = 1 - (1 - e^{-\lambda\times 1}) = e^{-\lambda\times 1}$. We need to find $\lambda > 0$ such that $\mathbb{P}(Y = 0) \geq 0.95$ We have

$$\begin{aligned}\mathbb{P}(Y=0) &= \sum_{n=0}^{\infty} \mathbb{P}(Y=0|N=n)\,\mathbb{P}(N=n) = \sum_{n=0}^{\infty} \binom{n}{0} q^0 (1-q)^n\, e^{-10} 10^n/n! \\ &= e^{-10} \sum_{n=0}^{\infty} (10(1-q))^n/n! = e^{-10} e^{10(1-q)} \\ &= e^{-10q}\end{aligned}$$

   Then $\mathbb{P}(Y = 0) \geq 0.95$ if and only if $-10 \times q \geq \log(0.95)$, then $q \leq -\log(0.95)/10$. Hence we need $e^{-\lambda} \leq -\log(0.95)/10$ or $\lambda \geq -\log(-\log(0.95)/10) \approx 5.27$.

2. If you took λ to denote **the expected value of** T(i.e. if $T \sim \mathcal{E}(1/\lambda)$):

   (a) Let T be the time it takes to connect a caller with a customer service representative (in hours). Then $T \sim \mathcal{E}(1/3)$. Waiting less than one hour is the event $\{T < 1\}$ which has probability $\mathbb{P}(T < 1) = F_T(1) = 1 - e^{-1/3} \approx 0.2834687$.

(b) Let N be the number of people calling the call centre in any one hour period, then $N \sim \mathcal{P}(10)$. Let Y be the number of customers calling in the next hour that have to wait more than hour to be connected. Then, if $T \sim \mathcal{E}(1/\lambda)$, then $Y|N = n \sim Bi(n, q)$ with $q = \mathbb{P}(T > 1) = 1 - (1 - e^{-1/\lambda}) = e^{-1/\lambda}$. We need to find $\lambda > 0$ such that $\mathbb{P}(Y = 0) \geq 0.95$ We have

$$\begin{aligned}
\mathbb{P}(Y = 0) &= \sum_{n=0}^{\infty} \mathbb{P}(Y = 0|N = n)\, \mathbb{P}(N = n) = \sum_{n=0}^{\infty} \binom{n}{0} q^0 (1-q)^n\, e^{-10} 10^n/n! \\
&= e^{-10} \sum_{n=0}^{\infty} (10(1-q))^n/n! = e^{-10} e^{10(1-q)} \\
&= e^{-10q}
\end{aligned}$$

Then $\mathbb{P}(Y = 0) \geq 0.95$ if and only if $-10 \times q \geq \log(0.95)$, then $q \leq -\log(0.95)/10$. Hence we need $e^{-1/\lambda} \leq -\log(0.95)/10$ or $1/\lambda \geq -\log(-\log(0.95)/10)$, which means $\lambda \leq \log(-10/\log(0.95)) \approx 0.1896533$

Problem 2

Let $Y = e^{-X}$ where $X \sim \mathcal{E}(\lambda)$, for some $\lambda > 0$.

(a) Find the value of λ the maximizes $V(Y)$

(b) What is $\lim_{\lambda \to \infty} V(Y)$?

Solution:

(a) Let $Y = e^{-X}$, since $V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$, we only need to find these two expected values. We have

$$\begin{aligned}
\mathbb{E}(Y) &= \int_0^{+\infty} \lambda\, e^{-t} e^{-\lambda\, t}\, dt = \lambda \int_0^{+\infty} e^{-t\,(\lambda+1)}\, dt \\
&= \frac{\lambda}{\lambda+1} \int_0^{+\infty} (\lambda+1)\, e^{-t\,(\lambda+1)}\, dt = \lambda/(\lambda+1)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}(Y^2) &= \int_0^{+\infty} \lambda\, e^{-2t} e^{-\lambda\, t}\, dt = \frac{\lambda}{\lambda+2} \int_0^{+\infty} (\lambda+2)\, e^{-t\,(\lambda+2)}\, dt \\
&= \lambda/(\lambda+2)
\end{aligned}$$

Thus, $V(Y) = \lambda/(\lambda+2) - (\lambda/(\lambda+1))^2$. We need to find the maximum of $h(\lambda) = \lambda/(\lambda+2) - (\lambda/(\lambda+1))^2$ for $\lambda > 0$ Note that $h(0) = 0$ and $\lim_{\lambda \to \infty} h(\lambda) = 1 - 1 = 0$. We have

$$h'(\lambda) = \frac{2}{(\lambda+2)^2} - 2\frac{\lambda}{(\lambda+1)^3}$$

and $h'(\lambda) = 0$ if and only if $\lambda(\lambda+2)^2 = (\lambda+1)^3$, which has solutions $\lambda = (-1 \pm \sqrt{5})/2$. Since $\lambda > 0$, the unique zero of $h'(\lambda)$ for $\lambda > 0$ is $\lambda = (-1+\sqrt{5})/2$, and this value maximizes $V(Y)$.

(b) $V(Y) = \lambda/(\lambda+2) - (\lambda/(\lambda+1))^2$ which has $\lim_{\lambda\to\infty} V(Y) = 1 - 1 = 0$.

Problem 3

Let (X, Y) have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 8\,x\,y & \text{if } 0 \le x \le a,\ x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find a so that the above is a valid pdf. Is it unique?

(b) Show that $\mathbb{E}(Y|X = x)$ is an increasing function of x.

Solution:

(a) Note that $0 \le x \le a,\ x \le y \le 1$ implies that $a \le 1$. Because if $a > 1$, then any point (x, y) with $1 < x \le y$ will necessarily have $y > 1$, and thus $f_{(X,Y)}(x, y) = 0$. So we must have $x \le 1$, and hence $a \le 1$.

If the support of the pdf is $0 \le x \le a,\ x \le y \le 1$ then we must have

$$1 = \int_0^a \int_x^1 8xy\,dy\,dx = 8\int_0^a x\left(\int_x^1 y\,dy\right)\,dx = 2a^2 - a^4 = a^2\,(2 - a^2)$$

Calling $u = a^2$ and finding the solutions to the quadratic equation $1 = u(2-u)$ we find that the only solution is $u = 1$, and hence, the a should satisfy $a^2 = 1$, and $a > 0$, so it has to be $a = 1$. Yes, it is unique.

(b) To find $\mathbb{E}[Y|X = x]$ we need to find $f_{Y|X=x}(y) = f_{(X,Y)}(x,y)/f_X(x)$. We have, if $x \in (0, 1)$:

$$\begin{aligned} f_X(x) &= \int_x^1 8xy\,dy = 8\,x\left(\left.\frac{y^2}{2}\right|_x^1\right) \\ &= 4\,x\,(1 - x^2) \end{aligned}$$

and $f_X(x) = 0$ if $x \notin (0, 1)$. Hence, if $x \in (0, 1)$

$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{(1-x^2)} \qquad x \le y \le 1$$

and $f_{Y|X=x}(y) = 0$ if $x \notin (0, 1)$ or $y \notin (x, 1)$. Thus:

$$E[Y|X = x] = \int_x^1 y\,f_{Y|X=x}(y)\,dy = \int_x^1 \frac{2y^2}{(1-x^2)}\,dy = \frac{2}{3}\left(\frac{1-x^3}{1-x^2}\right)$$

It is sufficient to show that $h(x) = (1 - x^3)/(1 - x^2)$ is increasing for $x \in (0, 1)$. We have

$$h'(x) = x(x^3 - 3x + 2)/(1 - x^2)^2$$

We need to show that $h'(x) > 0$ for $x \in (0, 1)$. It is sufficient to show that $h_2(x) = (x^3 - 3x + 2) > 0$. Note that $h_2(0) = 2$, $h_2(1) = 0$, and $h_2'(x) = 3x(x - 1) < 0$, and thus $h_2(x) > 0$ for $x \in (0, 1)$. This shows that $h'(x) > 0$, which implies the monotonicity of $E[Y|X = x]$.

Problem 4

Let $X \sim \mathcal{E}(\lambda)$, and $Y|X = x \sim \mathcal{U}(0, x)$.

(a) Find $\mathbb{E}[Y]$.

(b) Using a computer if necessary, find $\mathbb{P}(Y \geq 1)$ when $\lambda = 2$.

Solution:

(a) We will use that $\mathbb{E}[Y] = \mathbb{E}[\,\mathbb{E}[Y|X]]$. Note that $\mathbb{E}[Y|X = x] = x/2$, thus $\mathbb{E}[\,\mathbb{E}[Y|X]] = \mathbb{E}[X/2] = \mathbb{E}[X]/2 = 1/(2\,\lambda)$.

(b) Note that

$$f_{(X,Y)}(x, y) = \frac{\lambda}{x} e^{-\lambda\, x} \qquad 0 \leq x, \quad 0 \leq y \leq x$$

A naive approach would be to use that

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{(X,Y)}(x, y) = \int_{y}^{+\infty} \frac{\lambda}{x} e^{-\lambda\, x}\, dx$$

and, with $\lambda = 2$, try to calculate

$$\mathbb{P}(Y \geq 1) = \int_1^{+\infty} f_Y(y)\, dy = \int_1^{+\infty} \left(\int_y^{+\infty} \frac{2}{x} e^{-2\, x}\, dx \right) dy$$

However, it seems difficult to find an analytical expression for $f_Y(y)$ (the inner integral in the last expression). Using a computer to approximate a double integral is less effective than to approximate a single integral. So we flip the order of integration to express the above double integral as a single one (see Figure 1)

$$\mathbb{P}(Y \geq 1) = \int_1^{+\infty} \left(\int_1^{x} \frac{2}{x} e^{-2\, x}\, dy \right) dx = \int_1^{+\infty} \frac{2(x-1)}{x} e^{-2\, x}\, dx$$

which is fairly simple to calculate approximately. Using any scientific computing environment you get

$$\mathbb{P}(Y \geq 1) \approx 0.03753$$

For example, in `R` we get

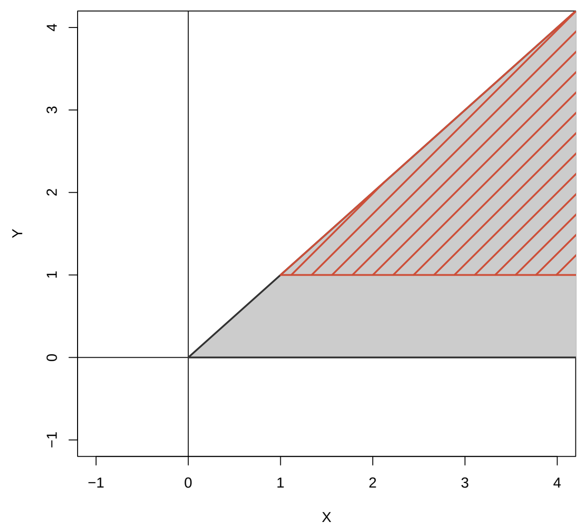

Figure 1: **Problem 4**: The gray area is the region where $f_{(X,Y)}(x, y) > 0$, and the shaded area is the region where $\{Y \geq 1\}$

```
> integrate(f=function(x, lambda) lambda*(x-1)/x*exp(-lambda*x),
+           lower=1, upper=+Inf, lambda=2)
0.03753426 with absolute error < 3.1e-05
```

Problem 5

Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x, y) = e^{-x}, \quad 0 < y < x < \infty$$

What is $Corr(X, Y)$?

Solution: To compute $Corr(X, Y)$ we need $Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$ and $V(X)$ and $V(Y)$. We first calculate the pdf of X and Y. We have

$$f_X(x) = \int_0^x e^{-x}\, dy = xe^{-x} \quad 0 < x$$

and $f_X(x) = 0$ if $x < 0$. Then,

$$\mathbb{E}[X] = \int_0^{+\infty} x\, f_X(x)\, dx = \int_0^{+\infty} x^2\, e^{-x}\, dx$$

which is $\mathbb{E}[W^2]$ with $W \sim \mathcal{E}(1)$. Since the variance of an $\mathcal{E}(1)$ random variable is 1, and its mean is also 1, we have that $\mathbb{E}[W^2] = V(W) + (\mathbb{E}[W])^2 = 1 + 1 = 2$. Hence

$$\mathbb{E}[X] = 2$$

Similarly, we have

$$\mathbb{E}[X^2] = \int_0^{+\infty} x^3\, e^{-x}\, dx$$

which is $\mathbb{E}[W^3]$ with $W \sim \mathcal{E}(1)$. It is easy to see that, in general

$$\int_0^{+\infty} x^n\, e^{-x}\, dx = n!$$

(prove it by induction, for example). Thus

$$\mathbb{E}[X^2] = 3! = 6$$

Finally, $V(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 6 - 4 = 2$

For Y we have, for $0 < y$:

$$f_Y(y) = \int_y^{+\infty} e^{-x}\, dx = -e^{-x}\Big|_y^{+\infty} = e^{-y}$$

and $f_Y(y) = 0$ for $y < 0$. Thus $Y \sim \mathcal{E}(1)$, and $\mathbb{E}[Y] = V(Y) = 1$.

Finally,

$$\begin{aligned}\mathbb{E}[XY] &= \int_0^{+\infty} \int_0^{x} xye^{-x}\, dy\, dx \\ &= \int_0^{+\infty} x\, e^{-x} \int_0^{x} y\, dy\, dx \\ &= \frac{1}{2} \int_0^{+\infty} x^3\, e^{-x}\, dx = \frac{1}{2} 3! = 3\end{aligned}$$

and

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)}\,\sqrt{V(Y)}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]}{\sqrt{2}\,\sqrt{1}} = \frac{3 - 2 \times 1}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Problem 7

Let $X \sim \mathcal{U}(0,1)$.

(a) Show that for any fixed $a \in (0,1)$, $h(k) = \mathbb{P}(X^{1/k} \leq a)$ is a decreasing function of $k \in \mathbb{N}$.

(b) Find k_0 such that for any $k \geq k_0$ we have $\mathbb{P}(X^{1/k} > 3/4) > 3/4$

Solution:

(a) We have

$$\mathbb{P}(X^{1/k} \le a) = \mathbb{P}(X \le a^k) = \int_0^{a^k} dx = a^k$$

which is a decreasing function of k if $a \in (0,1)$.

(b) We need to find k's such that

$$\begin{aligned} \mathbb{P}(X^{1/k} \ge 3/4) &\ge 3/4 \\ 1 - \mathbb{P}(X^{1/k} \le 3/4) &\ge 3/4 \\ 1 - (3/4)^k &\ge 3/4 \\ 1/4 &\ge (3/4)^k \\ \frac{\log(1/4)}{\log(3/4)} &\le k \end{aligned}$$

and hence, $k \ge 4.818842$, and since $k \in \mathbb{N}$ we have $k \ge 5 = k_0$.

Problem 8

Let X, Y be two random variables such that their joint PDF is given by $f_{(X,Y)}(x,y) = g(y - a - bx)f_X(x)$ for $x, y \in \mathbb{R}$, where a and b are fixed real numbers, $g : \mathbb{R} \to \mathbb{R}$ is a function that satisfies $\int_{-\infty}^{+\infty} g(t)\,dt = 1$ and $\int_{-\infty}^{+\infty} t\,g(t)\,dt = 0$, and f_X denotes the PDF of X. Show that $\mathbb{E}[Y|X] = a + b\,X$.

Solution: Since $f_{(X,Y)}(x,y) = g(y-a-bx)f_X(x)$ we have

$$f_{Y|X=x}(y) = \frac{f_{(X,Y)}(x,y)}{f_X(x)} = \frac{g(y-a-bx)f_X(x)}{f_X(x)} = g(y-a-bx)$$

Hence

$$\mathbb{E}[Y|X=x] = \int_{-\infty}^{+\infty} y\, f_{Y|X=x}(y)\,dy = \int_{-\infty}^{+\infty} y\, g(y-a-bx)\,dy$$

Since a, b and x are fixed constants in the above expression, let $t = y - a - bx$, then $dt = dy$, $y = t + a + bx$ and the limits of integration remain the same, thus

$$\begin{aligned} \mathbb{E}[Y|X=x] &= \int_{-\infty}^{+\infty} (t + a + bx)\, g(t)\,dt = \\ &= \int_{-\infty}^{+\infty} t\, g(t)\,dt + \int_{-\infty}^{+\infty} a\, g(t)\,dt + \int_{-\infty}^{+\infty} bx\, g(t)\,dt = \\ &= 0 + a\int_{-\infty}^{+\infty} g(t)\,dt + b\,x\int_{-\infty}^{+\infty} g(t)\,dt = a + b\,x \end{aligned}$$

Problem 9

Let X and Y be independent random variables, each uniformly distributed on $[0,1]$. Find $\mathbb{P}(\min\{2X+Y,\, X+2Y\} \le 1)$.

Solution: Note that the event $\{\min\{2X+Y,\, X+2Y\} \le 1\}$ is the same as

$$\{\min\{2X+Y,\, X+2Y\} \le 1\} \;=\; \{2X+Y \le 1\} \cup \{X+2Y \le 1\}$$

To prove this, note that if the right hand side union of events occurs, then either $2X+Y \le 1$, in which case $\min\{2X+Y,\, X+2Y\} \le 2X+Y \le 1$, or $X+2Y \le 1$ and then $\min\{2X+Y,\, X+2Y\} \le X+2Y \le 1$. Thus the right hand side is included in the event on the left hand side. To prove the other inclusion, note that if it were not included, then it would be included in the complement of the union, so that $2X+Y > 1$ and $X+2Y > 1$. But then we would have $\min\{2X+Y,\, X+2Y\} > 1$, which is a contradiction.

To simplify the notation, let $A = \{2X+Y \le 1\}$ and $B = \{X+2Y \le 1\}$. Then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. Using a graph to find the limits of integration (see Figure 2) we have

$$\mathbb{P}(A) = \mathbb{P}(2X+Y \le 1) = \int_0^{1/2} \int_0^{1-2x} dy\, dx = \int_0^{1/2} (1-2x)\, dx = 1/4$$

Similarly,

$$\mathbb{P}(B) = \mathbb{P}(X+2Y \le 1) = \int_0^{1} \int_0^{(1-x)/2} dy\, dx = \frac{1}{2} \int_0^{1} (1-x)\, dx = 1/4$$

Also using the graph in Figure 2, we find the region where both $2X+Y \le 1$ and $X+2Y \le 1$, and we break the integral into two parts:

$$\mathbb{P}(A \cap B) = \int_0^{1/3} \int_0^{(1-x)/2} dy\, dx + \int_{1/3}^{1/2} \int_0^{1-2x} dy\, dx =$$

$$= \frac{1}{2} \int_0^{1/3} (1-x)\, dx + \int_{1/3}^{1/2} (1-2x)\, dx = 1/6$$

Finally

$$\mathbb{P}(\min\{2X+Y,\, X+2Y\} \le 1) = \mathbb{P}(2X+Y \le 1) + \mathbb{P}(X+2Y \le 1) -$$
$$- \mathbb{P}(2X+Y \le 1,\, X+2Y \le 1) = 1/4 + 1/4 - 1/6 = 1/2 - 1/6 = 1/3$$

Problem 10

Let (X,Y) have joint pdf

$$f_{(X,Y)}(x,y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathbb{P}(Y - 2\,X < 0)$.

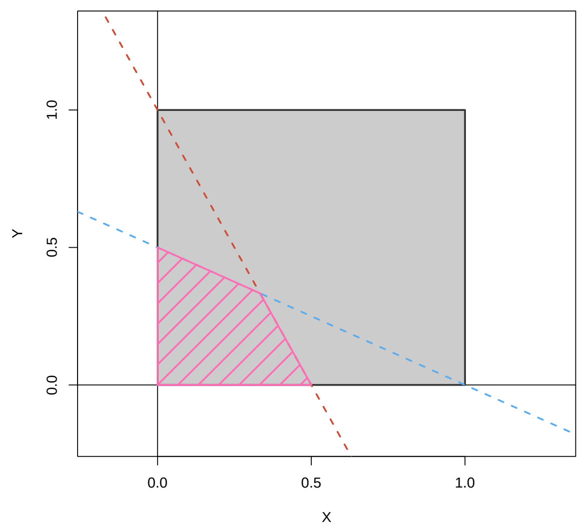

Figure 2: **Problem 9**: The gray area is the region where $f_{(X,Y)}(x, y) > 0$. The blue dashed line is $X + 2Y = 1$, the red dashed line is $Y + 2X = 1$, and the shaded region (pink) is the intersection of the regions below the two dashed lines.

Solution: A graph shows that the region of interest is the intersection between the upper triangle of $[0, 1] \times [0, 1]$ over the line $Y = X$, and the region below the line $Y = 2X$ (see Figure 3). There are two ways of finding the integral of $f_{(X,Y)}(x, y)$ over that region: calculate it directly, or compute the complement, which is the region above the line $Y = 2X$. For the former we have

$$\mathbb{P}(Y - 2\,X < 0) = \int_0^1 \int_{y/2}^{y} 2\,dx\,dy \;=\; 2\int_0^1 y/2\,dy \;=\; 1/2$$

while for the latter

$$\mathbb{P}(Y - 2\,X < 0) = 1 - \mathbb{P}(Y - 2\,X > 0) = 1 - \mathbb{P}(Y > 2X) =$$

$$= 1 - \int_0^{1/2} \int_{2x}^{1} 2\,dy\,dx = 1 - 2\int_0^{1/2} (1 - 2x)\,dx = 1 - 2\left(x - x^2\,\Big|_0^{1/2}\right) = 1/2$$

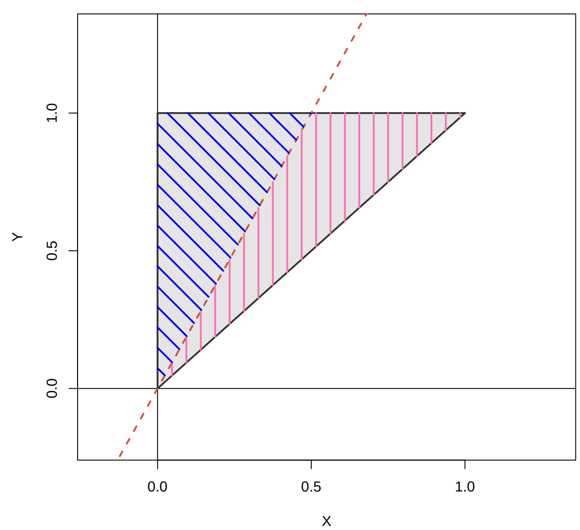

Figure 3: **Problem 10**: The gray area is the region where $f_{(X,Y)}(x, y) > 0$. The red dashed line is $Y = 2X$. The pink shaded region is $\{Y < 2X\}$. The blue shaded region is the region $\{Y > 2X\}$.