STAT302 Calculus Practice

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This note has a few questions to help you practice mathematical concepts from calculus that may appear on STAT302 exams.

Q. Evaluate $\lim_{n\to\infty} \exp(3 + 4/n)$.

Solution: By continuity of exp and addition, $\lim_{n\to\infty} \exp(3 + 4/n) = \exp(3 + \lim_n 4/n) = \exp(3)$.

Q. Evaluate $\lim_{x\to 0} \int_0^y \exp(3xt)\mathrm{d}t$.

Solution: If you start with the integral, you'll get $\lim_{x\to 0} \big((3x)^{-1}\exp(3xt)\big)_0^y = \lim_{x\to 0}(3x)^{-1}(\exp(3xy) - 1)$, and then use L'Hospital to find that $= \lim_{x\to 0}(3y\exp(3xy))/3 = y$.

Technically you can also do this by interchanging the limit and integral (but note that this isn't always allowed, and we don't expect you to know when this is allowed in this course) and use the continuity of exp: $\lim_{x\to 0}\int_0^y \exp(3xt)\mathrm{d}t = \int_0^y \lim_{x\to 0}\exp(3xt)\mathrm{d}t = \int_0^y \exp(0)\mathrm{d}t = y$.

Q. Evaluate $\lim_{x\to 0_+} \log\Big(\exp(3x - y) + \exp\Big(\log\Phi\big(\frac{1}{x^2}\big)\Big)\Big)$ (recall Φ is the standard normal CDF)

Solution: By continuity of log, exp, we can move the limit in:

$$\begin{aligned}
\lim(\dots) &= \log\left(\exp(3\lim_{x\to 0_+} x - y) + \exp\left(\log\lim_{x\to 0_+}\Phi\left(\frac{1}{x^2}\right)\right)\right)\\
&= \log\left(\exp(0 - y) + \exp\left(\log(\lim_{x\to\infty}\Phi(x))\right)\right)\\
&= \log(\exp(-y) + \exp(\log(1)))\\
&= \log(\exp(-y) + \exp(0))\\
&= \log(\exp(-y) + 1).
\end{aligned}$$

Q. Evaluate $\lim_{x\to\infty} \log\left(\frac{1}{1+e^{-x}}\right)$

Solution: By continuity of log,

$$\begin{aligned}
\lim(\dots) &= \log\left(\frac{1}{1 + \lim_{x\to\infty} e^{-x}}\right)\\
&= \log\left(\frac{1}{1+0}\right)\\
&= \log(1)\\
&= 0.
\end{aligned}$$

Q. Evaluate: $\lim_{x\to y}\int_0^\infty x^3\exp(-x^2 y)\mathrm{d}y$. Be careful here, this one is tricky on purpose.

Solution: This statement is *technically fine*, but has extremely bad notation. It's meant to stress test your understanding of dummy variables (I won't put something so evil on an exam). As long as you fully understand the meaning of a dummy variable of integration, you can work through it. The y outside the integral is some variable we haven't specified, so your answer could depend on it. The y in the integral is a dummy variable of integration; an equivalent but much clearer way to write this is to use a different name for the dummy variable:

$$\begin{aligned}\lim_{x\to y}\int_0^\infty x^3 \exp(-x^2 z)\mathrm{d}z &= \lim_{x\to y} x^3 \int_0^\infty \exp(-x^2 z)\mathrm{d}z \\ &= \lim_{x\to y} x^3 \frac{1}{x^2} \\ &= \lim_{x\to y} x = y.\end{aligned}$$

Q. Evaluate $\lim_{n\to\infty}\int_{-n}^n \exp(-(x-5)^2/2)\mathrm{d}x$.

Solution: Since the integrand doesn't depend on n, this is just the definition of the integral on the real line:

$$\lim_{n\to\infty}\int_{-n}^n \exp(-(x-5)^2/2)\mathrm{d}x. = \int \exp(-(x-5)^2/2)\mathrm{d}x.$$

The integrand is the Gaussian density centered at 5 with variance 1, so we can use the kernel integration trick:

$$= \sqrt{2\pi}\int \frac{1}{\sqrt{2\pi}}\exp(-(x-5)^2/2)\mathrm{d}x = \sqrt{2\pi}.$$

Q. Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\sum_{x=1}^5 f(x)g(5,x)$. Be careful, this is again tricky on purpose.

Solution: Once again, x is being used as a dummy variable in the sum *and* as a regular variable in the derivative. Again extremely bad notation that you hopefully won't find in the wild. But I'm stress testing your ability to distinguish dummy variables and non-dummy variables (I won't put something so evil on an exam). The answer is 0, since the statement is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{x=1}^5 f(x)g(5,x) = \frac{\mathrm{d}}{\mathrm{d}x}(f(1)g(5,1)+f(2)g(5,2)+f(3)g(5,3)+f(4)g(5,4)+f(5)g(5,5)) = 0,$$

because there is no dependence on x in the argument of the derivative.

Q. Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\sum_{y=1}^5 f(y)g(x,y)$.

Solution: Now x does appear in the statement:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sum_{y=1}^5 f(y)g(x,y) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(1)g(x,1)+f(2)g(x,2)+f(3)g(x,3)+f(4)g(x,4)+f(5)g(x,5)) \\ &= f(1)\frac{\mathrm{d}g}{\mathrm{d}x}(x,1)+f(2)\frac{\mathrm{d}g}{\mathrm{d}x}(x,2)+f(3)\frac{\mathrm{d}g}{\mathrm{d}x}(x,3)+f(4)\frac{\mathrm{d}g}{\mathrm{d}x}(x,4)+f(5)\frac{\mathrm{d}g}{\mathrm{d}x}(x,5) \\ &= \sum_{y=1}^5 f(y)\frac{\mathrm{d}g}{\mathrm{d}x}(x,y).\end{aligned}$$

Q. Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\log(1+e^{-3x})$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\dots) = \frac{1}{1+e^{-3x}} \cdot e^{-3x} \cdot (-3) = \frac{-3e^{-3x}}{1+e^{-3x}}$$

Q. Show that $f(x) = x^2$ is convex.

Solution: $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 2 \geq 0$, so f is convex.

Q. Is $f(x) = \int_0^1 \exp(sx)\mathrm{d}s$ convex?

Solution: $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} \int_0^1 \exp(sx)\mathrm{d}s = \int_0^1 \frac{\mathrm{d}^2 \exp(sx)}{\mathrm{d}x^2}\mathrm{d}s = \int_0^1 s^2 \exp(sx)\mathrm{d}s \geq 0$, so f is convex.

Q. Is $f(x) = x^3$ concave, convex, or neither?

Solution: $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 6x$. This is positive when $x \geq 0$, but when $x < 0$ it's negative. Thefore it is not nonnegative everywhere or nonpositive everywhere, so it is neither concave or convex.

Q. Show that $f(x) = \lim_{n\to\infty} 3(1 + x/n)^n$ is convex.

Solution: $\lim_n (1 + x/n)^n = e^x$, and $\frac{\mathrm{d}^2 e^x}{\mathrm{d}x^2} = e^x \geq 0$ for all x. Therefore f is convex.

Q. Show that the CDF of $\text{Exp}(\lambda)$ is concave (negative convex).

Solution: $F_X(x) = 1 - e^{-\lambda x}$, so $\frac{\mathrm{d}^2 F_X}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda e^{-\lambda x}\right) = -\lambda^2 e^{-\lambda x} \leq 0$. Therefore F_X is concave.

Q. Let $f(x) = x^2 e^{-x}$ for $x \geq 0$. Find $f'(x)$ and determine the value(s) of x where $f'(x) = 0$.

Solution: Apply the product rule with $u = x^2$ and $v = e^{-x}$:

$$f'(x) = 2x\, e^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2) = x\, e^{-x}(2 - x).$$

Setting $f'(x) = 0$: since $e^{-x} > 0$ for all x, we need $x(2 - x) = 0$, giving $x = 0$ or $x = 2$.

Q. Find each of the following derivatives and simplify: $\frac{d}{dx}\left[\log(1 + x^2)\right]$, $\frac{d}{dx}\left[e^{-x^2/2}\right]$.

Solution: (a) Let $u = 1 + x^2$, so $\dfrac{d}{dx}[\log u] = \dfrac{u'}{u}$:

$$\frac{d}{dx}\left[\log(1 + x^2)\right] = \frac{2x}{1 + x^2}.$$

(b) Let $u = -x^2/2$, so $\dfrac{d}{dx}[e^u] = e^u \cdot u'$:

$$\frac{d}{dx}\left[e^{-x^2/2}\right] = e^{-x^2/2} \cdot (-x) = -x\, e^{-x^2/2}.$$

Q. Let $f(x, y) = e^{-(x+y)}$ for $x, y \geq 0$. Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Then verify that $\frac{\partial^2 f}{\partial x\, \partial y} = f(x, y)$.

Solution: Write $f(x, y) = e^{-x}e^{-y}$. Treating the other variable as a constant:

$$\frac{\partial f}{\partial x} = -e^{-(x+y)}, \qquad \frac{\partial f}{\partial y} = -e^{-(x+y)}.$$

Differentiating $\partial f/\partial x$ with respect to y:

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial y}\Big[-e^{-(x+y)}\Big] = e^{-(x+y)} = f(x, y).$$

Q. Find $\frac{d}{dx}\frac{x}{(1+x)^2}$ and simplify your answer fully.

Solution: Let $u = x$ and $v = (1+x)^2$, so $u' = 1$ and $v' = 2(1+x)$. By the quotient rule:

$$\frac{d}{dx}\left[\frac{x}{(1+x)^2}\right] = \frac{u'v - uv'}{v^2} = \frac{(1+x)^2 - x\cdot 2(1+x)}{(1+x)^4}.$$

Factor $(1+x)$ from the numerator:

$$= \frac{(1+x)\big[(1+x) - 2x\big]}{(1+x)^4} = \frac{1-x}{(1+x)^3}.$$

Q. For $f(x, y) = 6x$ on the region $\mathcal{B} = \{(x, y) : 0 \le x \le y \le 1\}$ compute the double integral $\int\int_{\mathcal{B}} f(x, y)dydx$.

Solution: The constraint $0 \le x \le y \le 1$ means x ranges over $[0, 1]$, and for each x, y ranges from x to 1:

$$\int_0^1\int_x^1 6x\,dy\,dx = \int_0^1 6x(1-x)\,dx = \int_0^1 (6x - 6x^2)\,dx = \Big[3x^2 - 2x^3\Big]_0^1 = 3 - 2 = 1.$$

Q. Compute $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$ using integration by parts.

Solution: Let $u = x^2$ and $dv = e^{-x}\,dx$, so $du = 2x\,dx$ and $v = -e^{-x}$:

$$\int_0^\infty x^2 e^{-x}\,dx = \Big[-x^2 e^{-x}\Big]_0^\infty + 2\int_0^\infty x\,e^{-x}\,dx = 0 + 2\cdot 1 = 2.$$

(The boundary term vanishes since $x^2 e^{-x} \to 0$ as $x \to \infty$, and equals 0 at $x = 0$.)

Q. Given the joint density $f(x, y) = e^{-y}$ for $0 \le x \le y < \infty$, find the marginal density of X and identify its distribution.

Solution:

$$f_X(x) = \int_x^\infty e^{-y}\,dy = \Big[-e^{-y}\Big]_x^\infty = 0 - (-e^{-x}) = e^{-x}, \quad x \ge 0.$$

Therefore $X \sim \text{Exp}(1)$.

Q. Let $f(x) = x(1-x)^2$ for $x \in [0, 1]$. Use calculus to find the mode, i.e., the value of x that maximizes $f(x)$.

Solution: Expand and differentiate:

$$f(x) = x(1-x)^2 = x - 2x^2 + x^3.$$

$$f'(x) = 1 - 4x + 3x^2 = (1-x)(1-3x).$$

Critical points: $x = 1$ (boundary) and $x = \frac{1}{3}$ (interior).

Check the second derivative:

$$f''(x) = -4 + 6x \implies f''\left(\tfrac{1}{3}\right) = -4 + 2 = -2 < 0.$$

Since $f''(1/3) < 0$, the function is concave at $x = \frac{1}{3}$, confirming a local maximum.

Q. Let $f(x) = x^2$. Is f monotone increasing, decreasing, or neither?

Solution: To check monotonicity, take the derivative. $\frac{\mathrm{d}f}{\mathrm{d}x} = 2x$, which takes both positive and negative values depending on whether x is negative or positive. Therefore f is neither increasing nor decreasing.

Q. Let $f(x) = x^3$. Is f monotone increasing, decreasing, or neither?

Solution: To check monotonicity of a differentiable function, take the derivative. $\frac{\mathrm{d}f}{\mathrm{d}x} = 3x^2 > 0$ when $x \neq 0$ and ≥ 0 when $x = 0$, so f is strictly monotone increasing (the derivative is positive everywhere except at isolated points where it's 0).

Q. Let $f(x) = \frac{e^{-x}}{1+e^{-x}}$. Is f monotone increasing, decreasing, or neither?

Solution: To check monotonicity of a differentiable function, take the derivative.

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{-e^{-x}(1+e^{-x}) - (-e^{-x})e^{-x}}{(1+e^{-x})^2} \\
&= \frac{-e^{-x} - e^{-2x} + e^{-2x}}{(1+e^{-x})^2} \\
&= \frac{-e^{-x}}{(1+e^{-x})^2}.
\end{aligned}$$

Since $e^{\cdots}$ is always positive, and $(\ldots)^2$ is always positive, the derivative of f is always negative so f is strictly monotone decreasing.

Q. Let $h(s) = \int_0^1 \exp(sx)\mathrm{d}x$. Is h monotone increasing, decreasing, or neither?

Solution: To check monotonicity, take the derivative. Since both $\exp(sx)$ and its derivative is continuous, we can differentiate under the integral: $\frac{\mathrm{d}h}{\mathrm{d}s} = \int_0^1 x\exp(sx)\mathrm{d}x$. You could work out the integral via integration by parts, but it's much easier to notice that $x\exp(sx) > 0$ everywhere on $[0,1]$ except at $x = 0$. So the integral of $x\exp(sx)$ over $x \in [0,1]$ is strictly greater than 0 for all s. Therefore h is strictly monotone increasing.

Q. Let $h(s) = \int_0^{s^2} (-\exp(sx))\mathrm{d}x$. On the set $s \geq 0$, is h monotone increasing, decreasing, or neither?

Solution: To check monotonicity of a differentiable function, take the derivative. Since s appears both in the limits and in the integrand, we need to use the Leibniz rule:

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}s} &= (-\exp(s \cdot s^2))\frac{\mathrm{d}s^2}{\mathrm{d}s} - (-\exp(s \cdot 0))\frac{\mathrm{d}0}{\mathrm{d}s} + \int_0^{s^2} (-x\exp(sx))\mathrm{d}x \\
&= -2s\exp(s^3) - \int_0^{s^2} x\exp(sx)\mathrm{d}x.
\end{aligned}$$

On the set $s \geq 0$, both of these terms are strictly negative except at the single point $s = 0$. Therefore the function is strictly monotone decreasing.