

Lecture 3

Properties of Probability and Finite Spaces

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Today's Learning Outcomes

By the end of this lecture, students are anticipated to be able to

- Prove four properties of the probability function
- Apply the properties to word problems
- Calculate probabilities of finite sample spaces
- Apply to multiplicative principle

Properties of the Probability Function

Let A and B denote arbitrary events, where Ω is the sample space.

- **Probability of the complement:** $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ ✖
- **Monotonicity:** $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$
- **Probability of the union:** $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ ✖
- **Boole's inequality:** $\mathbb{P}(\bigcup_{i=1}^m A_i) \leq \sum_{i=1}^m \mathbb{P}(A_i)$

Properties of the Probability Function

EXERCISE: PROOF OF PROBABILITY OF THE COMPLEMENT

Prove the probability of the complement: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

To do this, show that if $\mathbb{P}$ satisfies Axioms 1, 2, and 3, and A is an arbitrary event, then necessarily $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Hint: What is $A \cup A^c$?

Recall: $A \cup A^c = \Omega$

$$\mathbb{P}(\Omega) = 1$$

$$\mathbb{P}(A \cup A^c) = 1$$

$$\mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

$$\Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

Axiom 1

Since $A \cup A^c = \Omega$

since A, A^c are disjoint
(Axiom 3)

by rearranging

Properties of the Probability Function

Properties of the Probability Function

💡 EXAMPLE

Prove monotonicity: $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Hints:

- Show $B = (B \cap A) \cup (B \cap A^c)$; ✓
- You can use the fact that if $A \subseteq B$, then $A \cap B = A$. Since we know that $A \subseteq B$, then $B \cap A = A$

Step 1: Show $B = (B \cap A) \cup (B \cap A^c)$

$$B = B \cap \Omega$$

$$= B \cap (A \cup A^c)$$

$$= (B \cap A) \cup (B \cap A^c) \quad \text{distributive property}$$

Properties of the Probability Function

Now, we have:

- $B = (B \cap A) \cup (B \cap A^c)$

- $A \subset B \Rightarrow A \cap B = A$

Therefore, as $A \subset B$, let's rewrite B as

$$\begin{aligned} B &= (B \cap A) \cup (B \cap A^c) \\ &= A \cup (B \cap A^c) \quad \text{since } A \subset B \Rightarrow A \cap B = A \end{aligned}$$

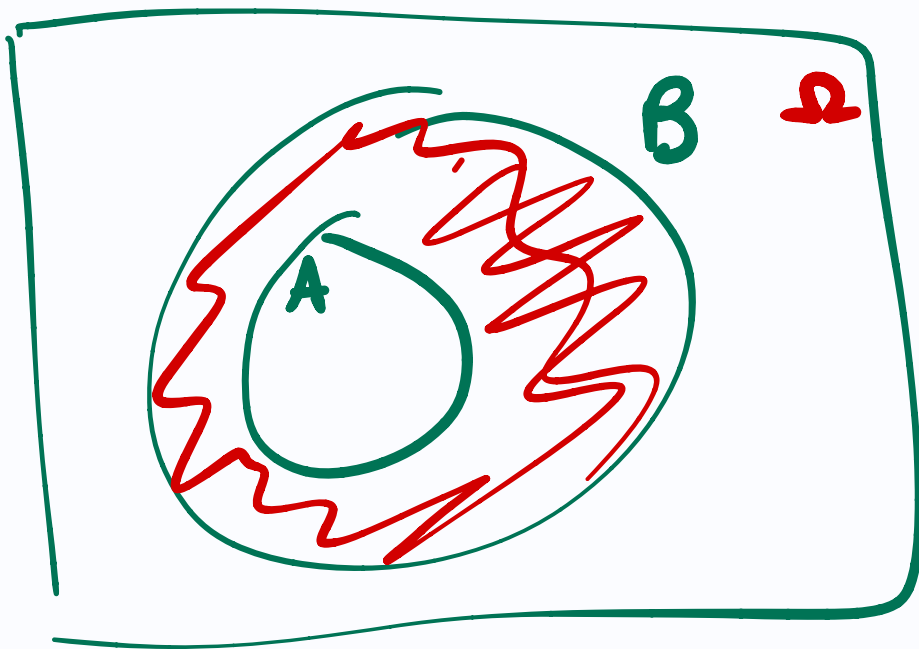
Then,

$$\begin{aligned} P(B) &= P[A \cup (B \cap A^c)] \\ &= P(A) + P(B \cap A^c) \quad \text{since } A \text{ and } B \cap A^c \text{ are disjoint} \end{aligned}$$

Properties of the Probability Function

Then, $P(B) = P[A \cup (B \cap A^c)]$
 $= P(A) + \underbrace{P(B \cap A^c)}_{\geq 0}$ since A and $B \cap A^c$ are disjoint
(by Axiom 2)

$$\Rightarrow P(B) \geq P(A)$$



Properties of the Probability Function

EXERCISE: PROOF OF PROBABILITY OF THE UNION

Prove the probability of the union: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

① Hint: First prove that $A \cup B = A \cup (B \cap A^c)$

Step 1: Show $A \cup B = A \cup (B \cap A^c)$

$$\begin{aligned} A \cup (B \cap A^c) &= (A \cup B) \cap (A \cup A^c) \quad (\text{distributive prop}) \\ &= (A \cup B) \cap \Omega \\ &= A \cup B \end{aligned}$$

Properties of the Probability Function

Step 2: Use the result of hint

$$A \cup B = A \cup (B \cap A^c)$$

$$\begin{aligned} \Rightarrow P(A \cup B) &= P[A \cup (B \cap A^c)] \\ &= P(A) + \underbrace{P(B \cap A^c)} \quad \text{since disjoint (Axiom 3)} \end{aligned}$$

Step 3 hint: can I express $P(B \cap A^c)$ in terms of $P(B)$ + something?

$$\begin{aligned} P(B) &= P(B \cap \Omega) \\ &= P[B \cap (A \cup A^c)] \\ &= P[(B \cap A) \cup (B \cap A^c)] \end{aligned}$$

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A^c) \quad (\text{disj}) \\ \Rightarrow \underbrace{P(B \cap A^c)} &= P(B) - P(B \cap A) \\ &\quad (\text{rearrange}) \end{aligned}$$

Properties of the Probability Function

Step 4: Put all together!

$$\begin{aligned} P(A \cup B) &= P(A) + P(B \cap A^c) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

OR do it in line!

Step 2: Use the result of line

$$\begin{aligned} \Rightarrow P(A \cup B) &= P[A \cup (B \cap A^c)] \\ &= P(A) + P(B \cap A^c) \\ &= P(A) + P(B \cap A^c) + P(A \cap B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

Properties of the Probability Function

💡 EXAMPLE

Prove Boole's inequality: $\mathbb{P}(\bigcup_{i=1}^m A_i) \leq \sum_{i=1}^m \mathbb{P}(A_i)$

$n=2$

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - \underbrace{P(A_1 \cap A_2)}_{\geq 0} \\ &\leq P(A_1) + P(A_2) \end{aligned}$$

$\therefore$ Boole's holds for $n=2$

Properties of the Probability Function

now, Assume this holds for $m \leq n$

$$P\left[\bigcup_{i=1}^{m+1} A_i\right] = P\left[\left(\bigcup_{i=1}^m A_i\right) \cup A_{m+1}\right]$$

assoc. prop

$$\leq P\left(\bigcup_{i=1}^m A_i\right) + P(A_{m+1})$$

from $n=2$

$$\leq \sum_{i=1}^m P(A_i) + P(A_{m+1})$$

by induction hyp.

$$= \sum_{i=1}^{m+1} P(A_i)$$

//

Applying Boole's

💡 EXAMPLE

Suppose that $\mathbb{P}(A) = 0.85$ and $\mathbb{P}(B) = 0.75$. Show that

$$\mathbb{P}(A \cap B) \geq 0.60.$$

$$P(A \cap B) = P[(A^c \cup B^c)^c]$$

$$= 1 - P(\underline{A^c \cup B^c})$$

$$\geq 1 - [P(A^c) + P(B^c)]$$

$$= 1 - [(1 - P(A)) + (1 - P(B))] \quad \text{comp rule}$$

$$= 1 - [(1 - 0.85) + (1 - 0.75)] = 0.60.$$

De Morgan's

comp-rule

Boole's

Aside!

Boole says:

$P(A^c \cup B^c) \leq P(A^c) + P(B^c)$
which implies

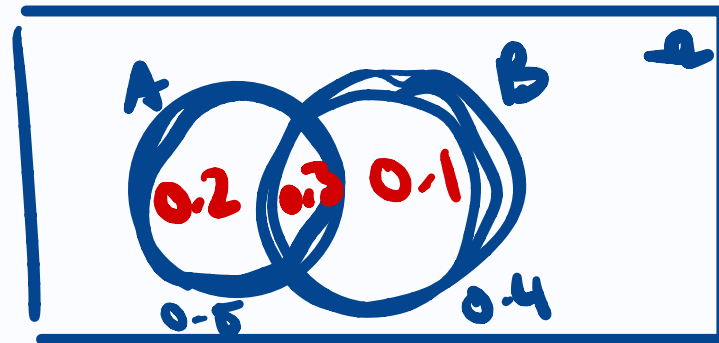
$$1 - P(A^c \cup B^c) \geq 1 - (P(A^c) + P(B^c))$$

Example

💡 EXAMPLE

Marley borrows 2 books. Suppose that there is a 0.5 probability they like the first book, 0.4 that they like the second book, and 0.3 that they like both.

What is the probability that they will **NOT** like **both** books? (i.e. that they will not like **either** book?)



A: Marley likes book 1
B: Marley likes book 2

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) && \text{D.M.} \\ &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] = 1 - (0.5 + 0.4 - 0.3) \\ &= 0.4 \end{aligned}$$

1 Uniform Probability on Finite Spaces

Finite and Equally Likely Outcomes

When there are **finitely many** outcomes, and they are **equal likely**, calculating probabilities involves counting outcomes in events/sets,

Let A be a subset (event) of a sample space Ω .

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

$$P(\text{flipping coin, landing on head}) = \frac{|\{H\}|}{|\{H, T\}|} = \frac{1}{2}$$

Finite and Equally Likely Outcomes

EXAMPLE

- Experiment: roll a fair die;
- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- If the die is **fair** we have

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\})$$

We have 6 possible outcomes, all equally likely.

Therefore the probability of rolling a any single number is $1/6$.

Finite and Equally Likely Outcomes

EXERCISE: THREE HEADS IN A ROW

Suppose that we flip three different fair coins. What is the probability of rolling three heads in a row?

$$\Omega = \{ \text{HHH}, \text{HHT}, \text{HTH}, \text{THH}, \text{HTT}, \text{TTH}, \text{THT}, \text{TTT} \}$$

$$|\Omega| = 2^3 = 8$$

$$A = \{ \text{HHH} \}$$

$$P(\{ \text{HHH} \}) = \frac{|A|}{|\Omega|} = \frac{1}{8}$$

Finite and Equally Likely Outcomes

- When the number of possible events are small, solving problems in this way is straightforward.
- However: counting the number possible events can be challenging, particularly as the number of possible events increases
 - Imagine rolling flipping 10 coins in a row. Writing out the sample space would be unreasonably long
- We will introduce permutations and combinations briefly to overcome this issue
 - These provide ways to count the number of possible events

Counting Sequences: Multiplicative Principle

If a random experiment has k steps.

- Step 1 has n_1 possible outcomes,
- Step 2 has n_2 possible outcomes,
- $\vdots$
- Step k has n_k possible outcomes.

Then,

$$\text{total number of outcomes} = n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Note

Implicit assumption: the outcomes of each step do not depend on each other.

Counting Sequences: Multiplicative Principle

💡 EXAMPLE → try at home ÷

Suppose that we flip three different fair coins. Without writing out the sample space, can you calculate the probability of rolling a head, a tail, and then a head?

Counting Sequences: Multiplicative Principle

EXERCISE: DRAWING 5 CARDS

What is the probability of drawing 5 cards in a row that are all clubs? Assume any card you picked is put back into the deck and shuffled before every draw.

We know $\frac{1}{4}$ of the deck is a club $\clubsuit$

$$P(\text{Club}) = \frac{1}{4}$$

$$P(\text{Draw 5 } \clubsuit \text{ in a row}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{1024}$$

2 Combinations

Combinations

DEFINITION

Combination (of size m): a subset of m items from a set of size n (where necessarily $m \leq n$).

Note: We only care which elements are in the set, not the ordering.

- Consider the set

$$S = \{a, b, c, d, e\}$$

- The following are all the possible subsets of S of size 3:

$\{a, b, c\}$	$\{a, d, e\}$
$\{a, b, d\}$	$\{b, c, d\}$
$\{a, b, e\}$	$\{b, c, e\}$
$\{a, c, d\}$	$\{b, d, e\}$
$\{a, c, e\}$	$\{c, d, e\}$

Combinations

- Note that **the order of the elements does not matter** (these are sets, not sequences).

$$\{a, b, d\} = \{d, a, b\} = \{b, d, a\}$$

(and other possible rearrangements).

Combinations

A useful mathematical property is the factorial.

DEFINITION

Factorial (!):

For any non-negative integer n :

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \dots 3 \times 2 \times 1$$

and

$$0! = 1$$

Number of Combinations

The number of combinations of size m out of a set of size $n \geq m$ has various notations:

$$\binom{n}{m} = {}_nC_m = C_m^n = \frac{n!}{m!(n-m)!}$$

- n : size of the set from which combinations are drawn
- m : size of the combinations
- we read it as “n choose m”



Tip

In this course we use $\binom{n}{m}$.

Example

There are five friends ($\{A, B, C, D, E\}$), and three concert tickets. How many different combinations of people can attend the concert?

Here, $n = 5$ and $m = 3$ we have

$$\begin{array}{cc} \{A, B, C\} & \{A, D, E\} \\ \hline \{A, B, D\} & \{B, C, D\} \\ \hline \{A, B, E\} & \{B, C, E\} \\ \hline \{A, C, D\} & \{B, D, E\} \\ \hline \{A, C, E\} & \{C, D, E\} \end{array}$$

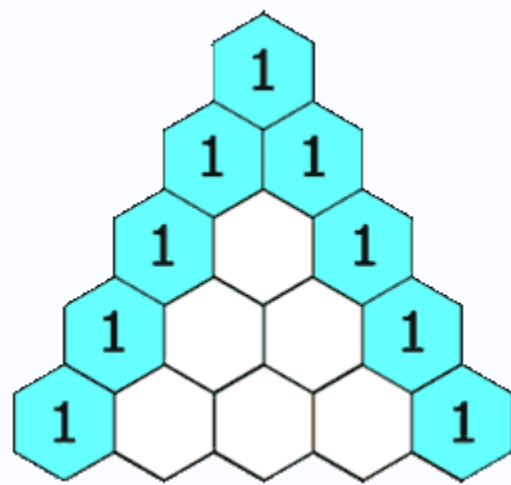
hence, the number of combinations must be

$$\binom{5}{3} = 10$$

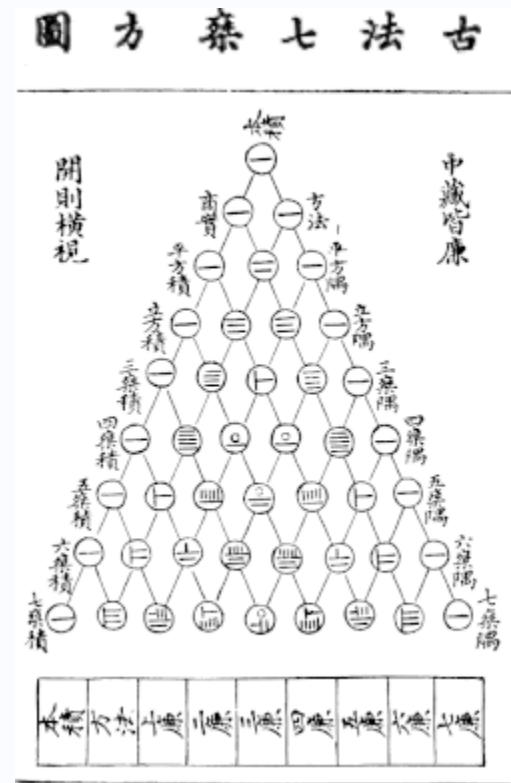
General formula

$$\binom{n}{m} = \frac{n!}{m!(n - m)!}$$

are also called **binomial coefficients** [↗](#).
They have many beautiful interpretations.



Pascal's Triangle



Yang Hui's Triangle

Checkpoint 1 is next week!

Your first checkpoint (one short question at the end of class ~ 15 minutes) is on Friday September 25th.

It is a small closed-book assessment based on the practice problems assigned for Sections 1.1 - 1.4 (all material so far up to the end of this lecture)

You may bring a **non-graphing, non-programmable calculator** and one **cheat sheet**, which you should add/edit to over time for all assessments. Cheat sheet rules:

- Must be 8.5 x 11 inches or smaller
- Can write on both sides
- MUST BE HANDWRITTEN, not photocopied or printed (not typed nor drawn on tablet)
- Write whatever you want on it

Cheat sheets and calculators that do not follow these rules will be confiscated - sorry!

CFA Students

- CFA students: you may choose to leave class when the quiz begins to write in the CFA, or you are welcome to stay in class to write.
- **It is your responsibility to book EVERY CHECKPOINT, midterm, and final exam with the CFA.** We do not have enough spaces/resources/TAs to provide accommodations in class, including extra time. There is a class directly after this one.