

Lecture 3

Properties of Probability and Finite Spaces

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Today's Learning Outcomes

By the end of this lecture, students are anticipated to be able to

- Prove four properties of the probability function
- Apply the properties to word problems
- Calculate probabilities of finite sample spaces
- Apply to multiplicative principle

Properties of the Probability Function

Let A and B denote arbitrary events, where Ω is the sample space.

- **Probability of the complement:** $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- **Monotonicity:** $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$
- **Probability of the union:** $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- **Boole's inequality:** $\mathbb{P}(\bigcup_{i=1}^m A_i) \leq \sum_{i=1}^m \mathbb{P}(A_i)$

Properties of the Probability Function

EXERCISE: PROOF OF PROBABILITY OF THE COMPLEMENT

Prove the probability of the complement: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

To do this, show that if $\mathbb{P}$ satisfies Axioms 1, 2, and 3, and A is an arbitrary event, then necessarily $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Hint: What is $A \cup A^c$?

Properties of the Probability Function

Properties of the Probability Function

EXAMPLE

Prove monotonicity: $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Hints:

- Show $B = (B \cap A) \cup (B \cap A^c)$;
- You can use the fact that if $A \subset B$, then $A \cap B = A$. 2. Since we know that $A \subset B$, then $B \cap A = A$

Properties of the Probability Function

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EXERCISE: PROOF OF PROBABILITY OF THE UNION

Prove the probability of the union: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Hint: First prove that $A \cup B = A \cup (B \cap A^c)$

Properties of the Probability Function

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EXAMPLE

Prove Boole's inequality: $\mathbb{P}(\bigcup_{i=1}^m A_i) \leq \sum_{i=1}^m \mathbb{P}(A_i)$

Properties of the Probability Function

Applying Boole's

EXAMPLE

Suppose that $\mathbb{P}(A) = 0.85$ and $\mathbb{P}(B) = 0.75$. Show that

$$\mathbb{P}(A \cap B) \geq 0.60.$$

Example

EXAMPLE

Marley borrows 2 books. Suppose that there is a 0.5 probability they like the first book, 0.4 that they like the second book, and 0.3 that they like both.

What is the probability that they will **NOT** like **both** books? (i.e. that they will not like **either** book?)

1 Uniform Probability on Finite Spaces

Finite and Equally Likely Outcomes

When there are **finitely many** outcomes, and they are **equal likely**, calculating probabilities involves counting outcomes in events/sets,

Let A be a subset (event) of a sample space Ω .

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

Finite and Equally Likely Outcomes

EXAMPLE

- Experiment: roll a fair die;
- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- If the die is **fair** we have

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\})$$

We have 6 possible outcomes, all equally likely.

Therefore the probability of rolling a any single number is $1/6$.

Finite and Equally Likely Outcomes

EXERCISE: THREE HEADS IN A ROW

Suppose that we flip three different fair coins. What is the probability of rolling three heads in a row?

Finite and Equally Likely Outcomes

- When the number of possible events are small, solving problems in this way is straightforward.
- However: counting the number possible events can be challenging, particularly as the number of possible events increases
 - Imagine rolling flipping 10 coins in a row. Writing out the sample space would be unreasonably long
- We will introduce permutations and combinations briefly to overcome this issue
 - These provide ways to count the number of possible events

Counting Sequences: Multiplicative Principle

If a random experiment has k steps.

- Step 1 has n_1 possible outcomes,
- Step 2 has n_2 possible outcomes,
- $\vdots$
- Step k has n_k possible outcomes.

Then,

$$\text{total number of outcomes} = n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Note

Implicit assumption: the outcomes of each step do not depend on each other.

Counting Sequences: Multiplicative Principle

EXAMPLE

Suppose that we flip three different fair coins. Without writing out the sample space, can you calculate the probability of rolling a head, a tail, and then a head?

Counting Sequences: Multiplicative Principle

EXERCISE: DRAWING 5 CARDS

What is the probability of drawing 5 cards in a row that are all clubs? Assume any card you picked is put back into the deck and shuffled before every draw.

2 Combinations

Combinations

DEFINITION

Combination (of size m): a subset of m items from a set of size n (where necessarily $m \leq n$).

Note: We only care which elements are in the set, not the ordering.

- Consider the set

$$S = \{a, b, c, d, e\}$$

- The following are all the possible subsets of S of size 3:

$$\{a, b, c\} \quad \{a, d, e\}$$

$$\{a, b, d\} \quad \{b, c, d\}$$

$$\{a, b, e\} \quad \{b, c, e\}$$

$$\{a, c, d\} \quad \{b, d, e\}$$

$$\{a, c, e\} \quad \{c, d, e\}$$

Combinations

- Note that **the order of the elements does not matter** (these are sets, not sequences).

$$\{a, b, d\} = \{d, a, b\} = \{b, d, a\}$$

(and other possible rearrangements).

Combinations

A useful mathematical property is the factorial.

DEFINITION

Factorial (!):

For any non-negative integer n :

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \dots 3 \times 2 \times 1$$

and

$$0! = 1$$

Number of Combinations

The number of combinations of size m out of a set of size $n \geq m$ has various notations:

$$\binom{n}{m} = {}_nC_m = C_m^n = \frac{n!}{m!(n-m)!}$$

- n : size of the set from which combinations are drawn
- m : size of the combinations
- we read it as “n choose m”



Tip

In this course we use $\binom{n}{m}$.

Example

There are five friends ($\{A, B, C, D, E\}$), and three concert tickets. How many different combinations of people can attend the concert?

Here, $n = 5$ and $m = 3$ we have

$$\begin{array}{cc} \{A, B, C\} & \{A, D, E\} \\ \hline \{A, B, D\} & \{B, C, D\} \\ \hline \{A, B, E\} & \{B, C, E\} \\ \hline \{A, C, D\} & \{B, D, E\} \\ \hline \{A, C, E\} & \{C, D, E\} \end{array}$$

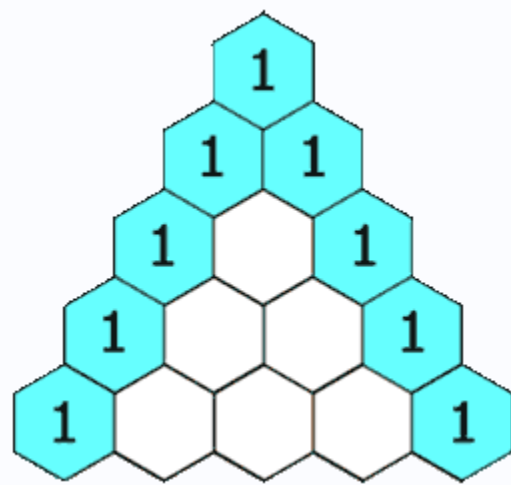
hence, the number of combinations must be

$$\binom{5}{3} = 10$$

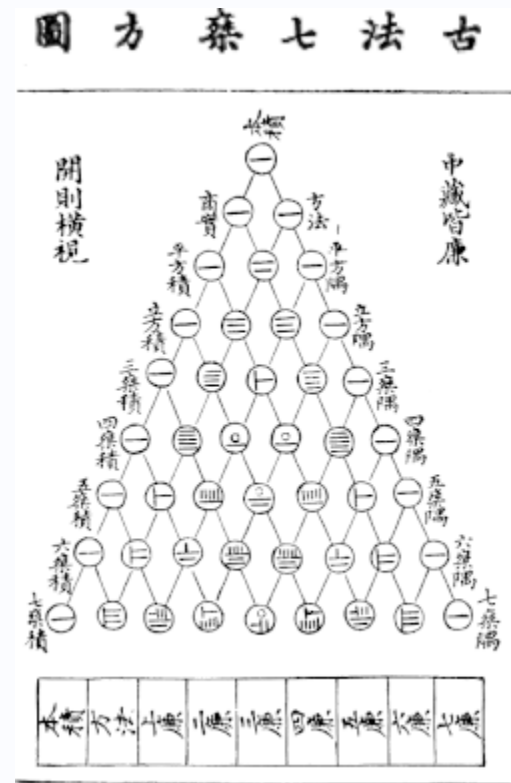
General formula

$$\binom{n}{m} = \frac{n!}{m!(n - m)!}$$

are also called **binomial coefficients** [↗](#).
They have many beautiful interpretations.



Pascal's Triangle



Checkpoint 1 is next week!

Your first checkpoint (one short question at the end of class ~ 15 minutes) is on Friday September 25th.

It is a small closed-book assessment based on the practice problems assigned for Sections 1.1 - 1.4 (all material so far up to the end of this lecture)

You may bring a **non-graphing, non-programmable calculator** and one **cheat sheet**, which you should add/edit to over time for all assessments. Cheat sheet rules:

- Must be 8.5 x 11 inches or smaller
- Can write on both sides
- MUST BE HANDWRITTEN, not photocopied or printed (not typed nor drawn on tablet)
- Write whatever you want on it

Cheat sheets and calculators that do not follow these rules will be confiscated - sorry!

CFA Students

- CFA students: you may choose to leave class when the quiz begins to write in the CFA, or you are welcome to stay in class to write.
- **It is your responsibility to book EVERY CHECKPOINT, midterm, and final exam with the CFA.** We do not have enough spaces/resources/TAs to provide accommodations in class, including extra time. There is a class directly after this one.