Lecture 3

Uniform Probability on Finite Spaces and Conditional Probabilities

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Last modified — 06 May 2026

Last Class:

- Defined probability as a mathematical object
- Used set-theory to compute probabilities and prove probabilistic properties

Today's Learning Outcomes

By the end of this lecture, students are anticipated to be able to

- Calculate the number of possible outcomes using:
  - Equally likely outcomes
  - Combinations , Permutation
- Define a conditional probability.

We will start with some word problems from last class' material

1 Applied Problems

Applied Problems

💡 EXAMPLE

Marley borrows 2 books. Suppose that there is a 0.5 probability they like the first book, 0.4 that they like the second book, and 0.3 that they like both.

What is the probability that they will **NOT** like **both** books? (i.e. that they will not like **either** book?)

A = {Marley likes book 1}
B = {Marley likes book 2}

$P(A) = 0.5$
$P(B) = 0.4$

$P(A \cap B) = 0.3$

A^c = {Marley doesn't like book 1}
B^c = {... book 2}

$$P(A^c \cap B^c) = P[(A \cup B)^c]$$ De Morgan

$$= 1 - P(A \cup B)$$ Comp. rule

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [0.5 + 0.4 - 0.3]$$

$$= 1 - 0.6 = 0.4$$

The probability that Marley will not like both books is 0.4

Applied Problems

✏️ EXERCISE: PASSING THE TEST

Jane must take two tests, call them T_1 and T_2. The probability that she passes test T_1 is 0.8, that she passes test T_2 is 0.7, and that of passing both tests is 0.6.

Calculate the probability that:

a. She passes at least one test. $P(A \vee B)$

b. She passes at most one test.

c. She fails both tests.

d. She passes only one test.

$A = \{\text{Jane passes } T_1\}$

$B = \{\text{Jane passes } T_2\}$

$P(A) = 0.8$

$P(B) = 0.7$

$P(A \cap B) = 0.6$

Applied Problems

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.8 + 0.7 - 0.6$

$= 0.9$

The probability that Jane passes at least one test is 0.9

b) P(pass at most one test)

= P(don't pass both)

$= P((A \cap B)^c)$

$= 1 - P(A \cap B)$ comp. rule

$= 1 - 0.6$

$= 0.4$

The probability that Jane passes at most one test is 0.4

Applied Problems

c) P(Jane fails both)

$= P(A^c \cap B^c)$

$= P[(A \cup B)^c]$ D.M.

$= 1 - P(A \cup B)$ comp. rule

$= 1 - 0.9$ part a)

$= 0.1$

The probability that Jane fails both tests is 0.1.

d) P(pass only one test)

$= P[(A \cap B^c) \cup (A^c \cap B)]$

$= P(A \cap B^c) + P(A^c \cap B)$ disjoint

$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$

$= 0.8 - 0.6 + 0.7 - 0.6$

$= 0.3$

The probability that Jane passes only one test is 0.3

Applied Problems

💡 EXAMPLE

Suppose that $\mathbb{P}(A) = 0.85$ and $\mathbb{P}(B) = 0.75$. Show that

$$\mathbb{P}(A \cap B) \geq 0.60.$$

$$P(A \cap B) = P[(A^c \cup B^c)^c]$$ De Morgan's

$$= 1 - P(A^c \cup B^c)$$ complement rule

$$\geq 1 - [P(A^c) + P(B^c)]$$ Boole's

$$= 1 - [(1 - P(A)) + (1 - P(B))]$$ complement rule

$$= 1 - [(1 - 0.85) + (1 - 0.75)]$$

$$= 0.60$$

Applied Problems

2 Uniform Probability on Finite Spaces

Finite and Equally Likely Outcomes

When there are **finitely many** outcomes, and they are **equal likely**, calculating probabilities involves counting outcomes in events/sets,

Let A be a subset (event) of a sample space Ω.

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

$$P(\text{Flip heads}) = \frac{|\{H\}|}{|\{H, T\}|} = \frac{1}{2}$$

Finite and Equally Likely Outcomes

💡 **EXAMPLE**

- Experiment: roll a fair die;
- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- If the die is **fair** we have

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\})$$

We have 6 possible outcomes, all equally likely.

Therefore the probability of rolling a any single number is $1/6$.

Finite and Equally Likely Outcomes

✏️ **EXERCISE: THREE HEADS IN A ROW**

Suppose that we flip three different fair coins. What is the probability of rolling three heads in a row?

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$|\Omega| = 2^3 = 8$

$P(\{HHH\}) = \frac{1}{8}$

Finite and Equally Likely Outcomes

- When the number of possible events are small, solving problems in this way is straightforward.
- However: counting the number possible events can be challenging, particularly as the number of possible events increases
  - Imagine rolling flipping 10 coins in a row. Writing out the sample space would be unreasonablly long
- We will introduce permutations and combinations briefly to overcome this issue
  - These provide ways to count the number of possible events

Counting Sequences: Multiplicative Principle

If a random experiment has **k steps**.

- Step 1 has n_1 possible outcomes,
- Step 2 has n_2 possible outcomes,

$$\vdots$$

- Step k has n_k possible outcomes.

Then,

$$\text{total number of outcomes} \;=\; n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Note

Implicit assumption: the outcomes of each step do not depend on each other.

Counting Sequences: Multiplicative Principle

💡 EXAMPLE

Suppose that we flip three different fair coins. Without writing out the sample space, can you calculate the probability of rolling a head, a tail, and then a head?

Sample space consists of (x_1, x_2, x_3) where $x_i = \{H, T\}$ $i=1,2,3$

Second flip will not depend on the first, third won't depend on the second so this is a 3-step experiment.

$$|\Omega| = 2 \cdot 2 \cdot 2 = 8$$

$$P(\{HTH\}) = \frac{1}{8}$$

Counting Sequences: Multiplicative Principle

✏️ **EXERCISE: DRAWING 5 CARDS**

What is the probability of drawing 5 cards in a row that are all clubs? Assume any card you picked is put back into the deck and shuffled before every draw.

1/4 of the deck is a club ♣.

$P(\text{club}) = 1/4$. $k = 5$ step experiment.

$$P(\text{draw 5 clubs in a row}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \left(\frac{1}{4}\right)^5$$

$$= \frac{1}{1024}$$

3 Combinations

Combinations

📖 DEFINITION

Combination (of size m): a subset of m items from a set of size n (where necessarily $m \leq n$).

Note: We only care which elements are in the set, not the ordering.

- Consider the set

$$S = \{a, b, c, d, e\}$$

n= 5

- The following are all the possible subsets of S of size 3: m=3

$\{a, b, c\}$	$\{a, d, e\}$
$\{a, b, d\}$	$\{b, c, d\}$
$\{a, b, e\}$	$\{b, c, e\}$
$\{a, c, d\}$	$\{b, d, e\}$
$\{a, c, e\}$	$\{c, d, e\}$

Combinations

- Note that the order of the elements does not matter (these are sets, not sequences).

$$\{a, b, d\} = \{d, a, b\} = \{b, d, a\}$$

(and other possible rearrangements).

Combinations

A useful mathematical property is the factorial.

📖 DEFINITION

Factorial (!):

For any non-negative integer n:

n(n-1) x (n-2) . . . . 3 x 2 x1

$$n! = (n-1) \times (n-2) \times (n-3) \dots 3 \times 2 \times 1$$

and

$$0! = 1$$

6! = 6x5x4x3x2 x1 = 720

Number of Combinations

The number of combinations of size m out of a set of size $n \geq m$ has various notations:

$$\binom{n}{m} = {}_nC_m = C_m^n = \frac{n!}{m!(n-m)!}$$

- n: size of the set from which combinations are drawn
- m: size of the combinations
- we read it as “n choose m”

Tip

In this course we use $\binom{n}{m}$.

Example

For example, when $n = 5$ and $m = 3$ we have

$\{1,2,3\}$	$\{1,4,5\}$
$\{1,2,4\}$	$\{2,3,4\}$
$\{1,2,5\}$	$\{2,3,5\}$
$\{1,3,4\}$	$\{2,4,5\}$
$\{1,3,5\}$	$\{3,4,5\}$

hence, the number of combinations must be

$$\binom{5}{3} = 10$$

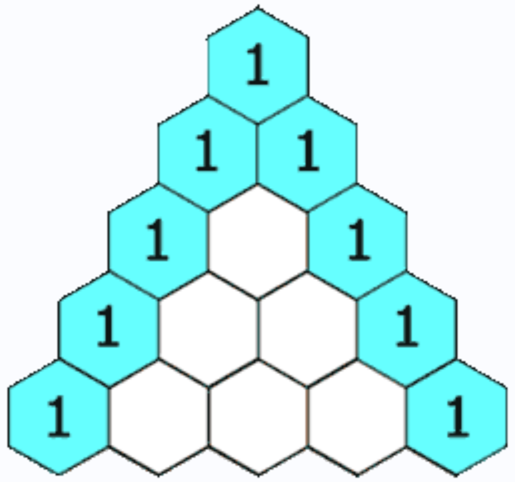
$$\frac{n!}{m!(n-m)!} = \frac{5!}{3!(5-3)!} = \frac{5\times4\times3\times2\times1}{3\times2\times1(2\times1)} = 10$$

General formula

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

are also called binomial coefficients.

They have many beautiful interpretations.



Pascal's Triangle

Yang Hui's Triangle

Lotto 6/49

- Dealer secretly chooses 6 distinct integers between 0 and 49.
- Player randomly selects 6 distinct integers between 0 and 49.
- The more matches, the bigger the prize.

> ✏️ **EXERCISE: LOTTO 6/49**
>
> What is the probability that the player matches k numbers from the dealer's selection (for different values of $k \in \{0, 1, \ldots, 6\}$)?

Number of possible combinations of size m=6 from n=50

$$\binom{50}{6} = \frac{50!}{6!(50-6)!} = 15890700$$

Lotto 6/49

K=1 (match one number)

$$\frac{\binom{6}{1}\binom{44}{5}}{\binom{50}{6}} = 0.444$$

match 1 of dealer's #

match 5 of numbers dealer didn't draw

K=2 match 2 of 6 number

$$\frac{\binom{6}{2}\binom{44}{4}}{\binom{50}{6}} = 0.410$$

Lotto 6/49

k=6

$$\frac{\binom{6}{6}\binom{44}{0}}{\binom{50}{6}} = 6.29 \times 10^{-8}$$

general for any k

$$\frac{\binom{6}{k}\binom{44}{6-k}}{\binom{50}{6}}$$

4 Permutations

Permutations

📖 **DEFINITION**

A permutation of a set is an arrangement of its elements in a specific order.

For example, calculating the number of ways:

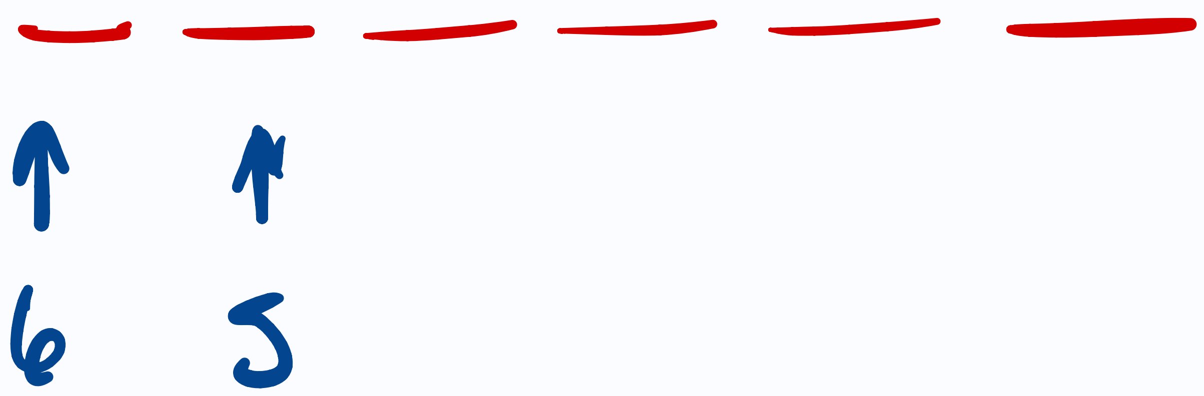
- 3 people can sit next to each other in 3 empty seats
- 10 people can be randomly assigned to three distinct volunteer positions

Suppose $|\Omega|$ = n and we want to count the number of permutations of length $k \leq n$ obtained from Ω.

In general, there are $n \times n - 1 \times \ldots \times n - k + 1$ permutations of length k from a set of n elements.

Permutations

> ✏️ **EXERCISE: REARRANGE SIX LETTERS**
>
> How many ways can the letters ABCDEF be rearranged?



$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Permutations

PEPPER

💡 EXAMPLE

In how many different ways can the letters of "pepper" be arranged?

If the letters were distinct → $6! = 720$ ways to rearrange

→ permute 3 p's : $3!$ ways

permute 2 e's : $2!$ ways

$$\Rightarrow \frac{6!}{3!\,2!} = 60$$

5 Conditional Probability

Conditional Probability

- In general, the outcome of a random experiment can be any element of Ω.
- Sometimes, we have "partial information" about which elements can occur.

> 💡 EXAMPLE
>
> - Roll a die.
> - If A is the event of obtaining a "2", then $\mathbb{P}(A) = 1/6$.
> - But if the outcome is known to be even, then intuition suggests that $\mathbb{P}(A) > 1/6$.

- Conditional probability formalizes this intuition (and helps to avoid mistakes)

Conditional Probability

- Two events play distinct roles in this example:
- The event of interest $A = \{2\}$
- The conditioning event

$$B = \{\text{outcome is even}\} = \{2, 4, 6\}$$

- The conditioning event captures the "partial information"
- If we only consider the three possible outcomes in B (even numbers), only one of them is a 2. Therefore, the probability of rolling a 2 if you know the roll was even is indeed 1/3.

Conditional Probability

The probability of an event A, conditional on event B is written as

$$\mathbb{P}(A \mid B)$$

We read this as "the probability of A *given* B" or "the probability of A *conditional on* B"

Conditional Probability (Formal definition)

- Let $A, B \subseteq \Omega$ and assume $\mathbb{P}(B) > 0$

📖 DEFINITION

- **The conditional probability of A given B is**

$$\mathbb{P}\left(A \mid B\right) = \frac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(B\right)}$$

- Just as $\mathbb{P}(\cdot)$ is a function, for any fixed B, $P\left(\ \cdot \ \mid B\right)$ is also a function. Its argument is any event $A \subseteq \Omega$.
- Moreover, $\mathbb{P}\left(\ \cdot \ \mid B\right)$ satisfies the three Axioms of a Probability (i.e., is a probability).

Useful Result:

As $P(\cdot \mid B)$ is a probability, for any event A:

- $\mathbb{P}(A \mid B) + \mathbb{P}(A^C \mid B) = 1$
- This implies:

$$\mathbb{P}(A^C \mid B) = 1 - \mathbb{P}(A \mid B)$$

we do not take B^c here

Formalizing our intuition

✏️ **EXERCISE: MATCHING COIN FLIPS**

Your friend flips two coins, looks at it, and tells you that the two faces are the same.

What is the probability that both coins show heads? Use the definition of conditional probability to solve this.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Event of interst: $A = \{HH\}$

Conditioning event: $B = \{HH, TT\}$

$\Omega = \{HH, TT, HT, TH\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = 1/2$$

Formalizing our intuition

~~Formalizing our intuition~~ Additional Practice

Textbook: 1.3.2

(b) What is the smallest possible value of $P(\{1, 2, 3\})$?

1.3.2 Suppose that Al watches the six o'clock news 2/3 of the time, watches the eleven o'clock news 1/2 of the time, and watches both the six o'clock and eleven o'clock news 1/3 of the time. For a randomly selected day, what is the probability that Al watches only the six o'clock news? For a randomly selected day, what is the probability that Al watches neither news?

$A = \{6 \text{ oclock news}\}$ $P(A) = 2/3$

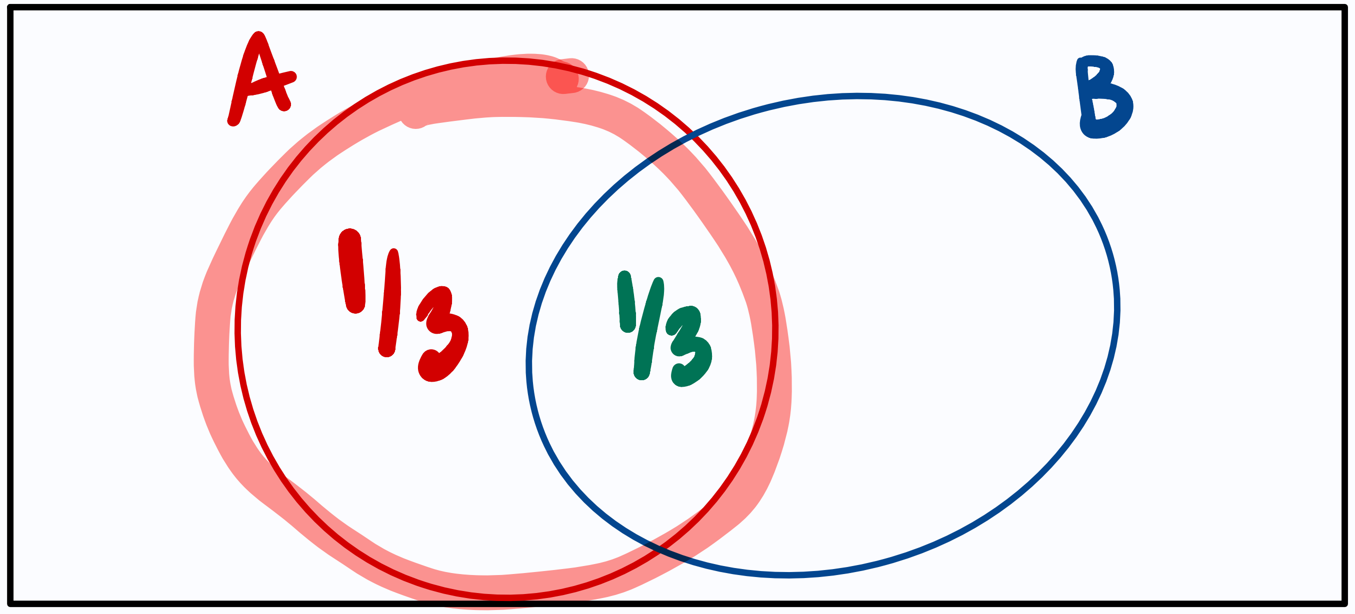
$B = \{11 \text{ oclock news}\}$ $P(B) = 1/2$

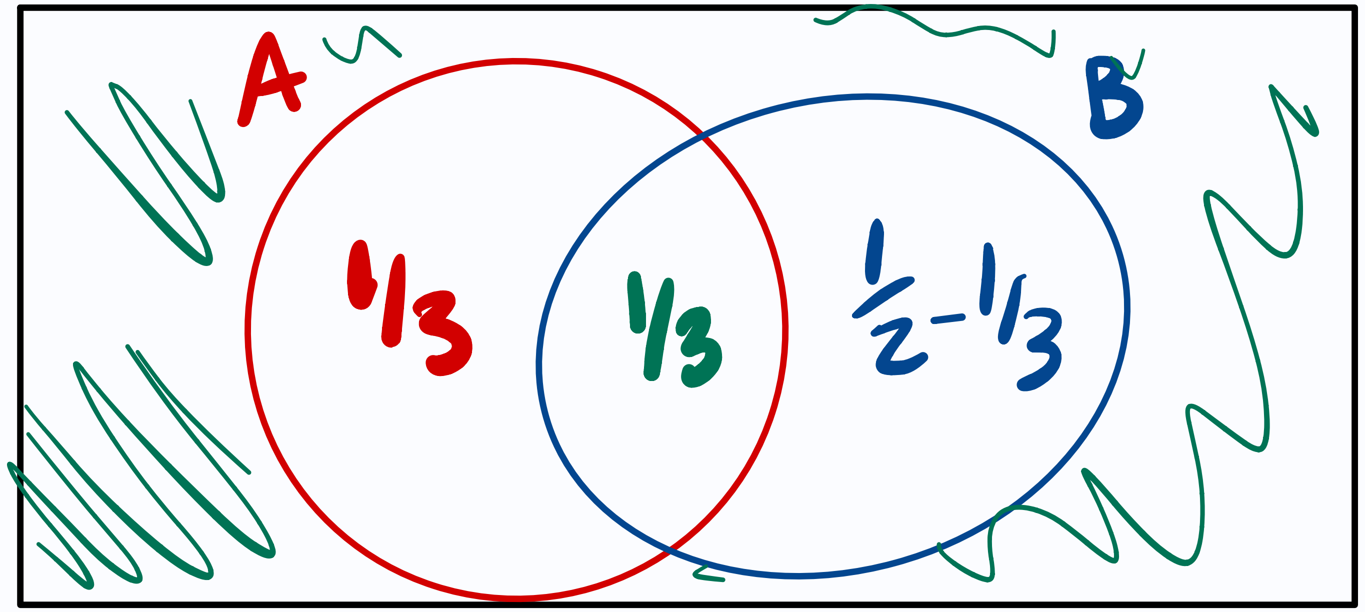
$P(A \cap B) = 1/3$

a) $P(A \cap B^c)$

b) $P(A^c \cap B^c)$

Formalizing our intuition





a)

$P(A) = P(A \cap B^c) \cup P(A \cap B)$

$2/3 = P(A \cap B^c) + 1/3$

$\therefore \Rightarrow P(A \cap B^c) = 1/3$

b) $P(A^c \cap B^c)$

$= P[(A \cup B)^c]$ De Morg

$= 1 - P(A \cup B)$ comp rule

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [2/3 + 1/2 - 1/3] = 1/6$

To Do:

- Assignment 1 due Wednesday May 20, 11:59pm
- Read Chapter 1.5.1 and 1.5.2 before Tuesday's class

Next Week

- More Conditional Probability, Independence
- Random Variables, Distributions
- Discrete Random Variables 🤠

Fix annotated slides