

Lecture 4

Conditional Probability (Part 1)

Grace Tompkins

Last modified — 13 Jul 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Use intuition to conditional probabilities from sets
- Calculate conditional probabilities using the definitions
- Use the law of total probability

1 Conditional Probability

Conditional Probability

- In general, the outcome of a random experiment can be any element of Ω .
- Sometimes, we have “partial information” about which elements can occur.

EXAMPLE

- Roll a die.
 - If A is the event of obtaining a “2”, then $\mathbb{P}(A) = 1/6$.
 - But if the outcome is **known** to be even, then intuition suggests that $\mathbb{P}(A) > 1/6$.
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- Conditional probability formalizes this intuition (and helps to avoid mistakes)

Conditional Probability

- Two events play distinct roles in this example:
- The event of interest $A = \{2\}$
- The conditioning event

$$B = \{\text{outcome is even}\} = \{2, 4, 6\}$$

- The conditioning event captures the “partial information”
- If we only consider the three possible outcomes in B (even numbers), only one of them is a 2. Therefore, the probability of rolling a 2 if you know the roll was even is indeed $1/3$.

Conditional Probability

The probability of an event A , conditional on event B is written as

$$\mathbb{P}(A | B)$$

We read this as “the probability of A *given* B ” or “the probability of A *conditional on* B ”

Event
of interest

Conditioning event

Conditional Probability (Formal definition)

- Let $A, B \subseteq \Omega$ and assume $\mathbb{P}(B) > 0$

DEFINITION

- The conditional probability of A given B is

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad \text{for } \mathbb{P}(B) > 0$$

- Just as $\mathbb{P}(\cdot)$ is a function, for any fixed B , $\mathbb{P}(\cdot | B)$ is also a function. Its argument is any event $A \subseteq \Omega$.
- Moreover, $\mathbb{P}(\cdot | B)$ satisfies the three Axioms of a Probability (i.e., is a probability).

Conditional Probability (Formal definition)

EXERCISE:

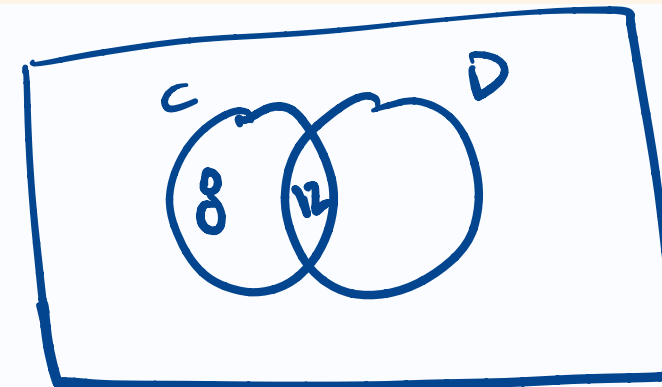
100 people were surveyed about their pets. 20 people had a cat, and 12 people had both a dog and a cat. What is the probability that a randomly selected person has a dog, given that they have a cat?

C: have a cat

D: have a dog

$$P(C) = 20/100$$

$$P(D \cap C) = 12/100$$



$$P(D | C) = \frac{P(D \cap C)}{P(C)} = \frac{12/100}{20/100} = \frac{12}{20} = \frac{3}{5}$$

Useful Result:

As $P(\cdot \mid B)$ is a probability, for any event A :

- $\mathbb{P}(A \mid B) + \mathbb{P}(A^C \mid B) = 1$
- This implies:

$$\mathbb{P}(A^C \mid B) = 1 - \mathbb{P}(A \mid B)$$

Formalizing our intuition

$$\Omega = \{HH, TT, HT, TH\}$$

EXERCISE: MATCHING COIN FLIPS

Your friend flips two coins, looks at it, and tells you that the two faces are the same.

What is the probability that both coins show heads? Use the definition of conditional probability to solve this.

$$A = \{HH\}$$

$$B = \text{two faces same} = \{HH, TT\}$$

↳ HH OR TT

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1/4}{2/4}$$

$$= 1/2$$

Formalizing our intuition

Multiplication Property

If $\mathbb{P}(A_1) > 0$:

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_2 \mid A_1) \mathbb{P}(A_1).$$

This is just a rearrangement of our definition!

COROLLARY

If $\mathbb{P}(A_1), \mathbb{P}(A_1 \cap A_2), \dots, \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_{n-1}) > 0$, then

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) &= \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \dots \cap A_{n-1}) \\ &\quad \times \mathbb{P}(A_{n-1} \mid A_1 \cap A_2 \cap \dots \cap A_{n-2}) \\ &\quad \times \dots \times \\ &\quad \times \mathbb{P}(A_3 \mid A_1 \cap A_2) \times \mathbb{P}(A_2 \mid A_1) \times \mathbb{P}(A_1) \end{aligned}$$

Proof of multiplication property

Consider $n=4$

$$P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot P(A_4 | A_1 \cap A_2 \cap A_3)$$

$$= \cancel{P(A_1)} \cdot \frac{\cancel{P(A_1 \cap A_2)}}{\cancel{P(A_1)}} \cdot \frac{\cancel{P(A_1 \cap A_2 \cap A_3)}}{\cancel{P(A_1 \cap A_2)}} \cdot \frac{P(A_1 \cap A_2 \cap A_3 \cap A_4)}{\cancel{P(A_1 \cap A_2 \cap A_3)}}$$

$$= P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

Urns and Balls

EXERCISE: URNS AND BALLS

- An urn has 10 red balls and 40 black balls.
- Three balls are randomly drawn without replacement.

Calculate the probability that:

- a. The 3rd ball is red given that the 1st is red and the 2nd is black. $P(R_3 | R_1 \cap B_2)$
- b. The first drawn ball is red, the 2nd is black and the 3rd is red. $P(R_1 \cap B_2 \cap R_3)$

$R_i = \{i^{\text{th}} \text{ ball is red}\}$
 $B_i = \{i^{\text{th}} \text{ ball is black}\}$

$$b) P(R_1 \cap B_2 \cap R_3) = P(R_1) \cdot P(B_2 | R_1) \cdot P(R_3 | R_1 \cap B_2) = \frac{10}{50} \cdot \frac{40}{49} \cdot \frac{9}{48} = 0.031$$

Urns and Balls

a) $P(R_3 | R_1 \cap B_2)$

↳ if the first ball is red → 9 red remaining

↳ if the 2nd is black → 39 black remaining

Then, on the third draw: $P(R_3 | R_1 \cap B_2) = \frac{9}{(39+9)} = \frac{9}{48}$

OR

$$P(R_3 | R_1 \cap B_2) = \frac{P(R_1 \cap B_2 \cap R_3)}{P(R_1 \cap B_2)} = \frac{\frac{10}{50} \cdot \frac{40}{49} \cdot \frac{9}{48}}{\frac{10}{50} \cdot \frac{40}{49}} = \frac{9}{48}$$

The “Total Probability” Formula

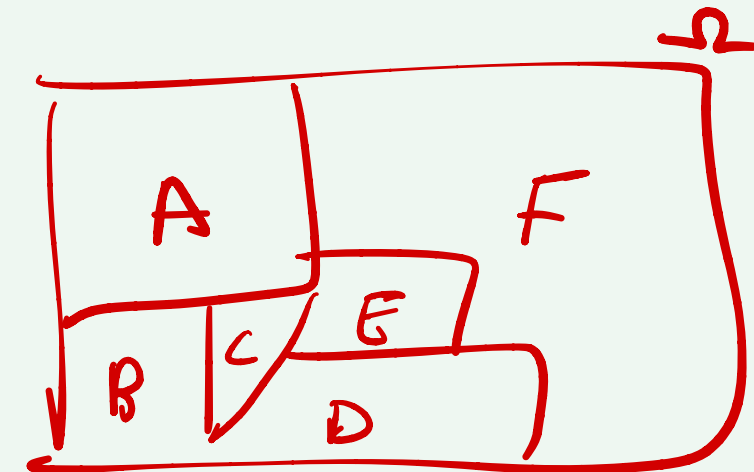
📖 DEFINITION

We say that $B_1, \dots, B_n$ is a **partition** of Ω if

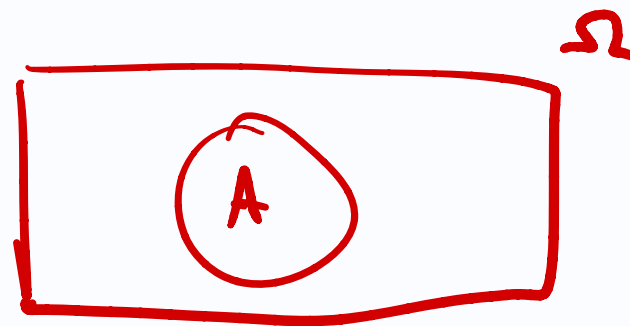
1. They are disjoint

$$B_i \cap B_j = \emptyset \quad \text{for } i \neq j,$$

2. They cover the whole sample space: $\bigcup_{i=1}^n B_i = \Omega$



A simple partition is any event A and its complement A^c .



The “Total Probability” Formula

THEOREM

If $B_1, \dots, B_n$ is a **partition** of Ω , then, for any $A \subset \Omega$,

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A \mid B_i) \mathbb{P}(B_i).$$

Example:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Proof of Total Probability

PROOF

- $A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n (A \cap B_i)$
- The events $(A \cap B_i)$ are disjoint.
- Therefore, by Axiom 3, we have

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P} \left(\bigcup_{i=1}^n A \cap B_i \right) \\ &= \sum_{i=1}^n \mathbb{P}(A \cap B_i) \\ &= \sum_{i=1}^n \mathbb{P}(A \mid B_i) \mathbb{P}(B_i). \end{aligned}$$

Flu Test

- Suppose that every patient who visits the ER is given a flu test.
- Suppose that 30% of patients have flu.
- A patient with flu tests positive 90% of the time.
- A patient without flu tests negative 80% of the time.

$$\begin{aligned}P(B|A^c) &= 1 - P(B^c|A^c) \\&= 1 - 0.8 \\&= 0.2\end{aligned}$$

EXERCISE: FLU TEST

If a new patient walks into the ER, what is the probability that they test positive for the flu?

$A = \{\text{patient has flu}\}$

$B = \{\text{test positive}\}$

$$P(A) = 0.3 \quad P(A^c) = 0.7$$

$$P(B|A) = 0.9$$

$$P(B^c|A^c) = 0.8$$

$$\begin{aligned}P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\&= (0.9)(0.3) + (0.2)(0.7) \\&= 0.41\end{aligned}$$

Checkpoint 1 is this Friday!

Your first checkpoint (one short question at the end of class ~ 15 minutes) is on Friday September 25th
It is a closed-book assessment based on the practice problems assigned for **Sections 1.1 - 1.4 (all material so far up to the end of this lecture)**

You may bring a **non-graphing, non-programmable calculator** and one **cheat sheet**, which you should add/edit to over time for all assessments. The cheat sheet must be:

- 8.5 x 11 inches or smaller
- Can write on both sides
- HANDWRITTEN, not photocopied or printed (typed or drawn on tablet)
- Write whatever you want on it

Cheat sheets and calculators that do not follow these rules will be confiscated - sorry!