

Lecture 4

Conditional Probability and Independence

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Last modified — 06 May 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a conditional probability
- Solve problems using conditional probability rules, including Bayes' theorem
- Identify when events are independent

1 Conditional Probability

Last class:

DEFINITION

- The conditional probability of A given B is

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$P(B) > 0$$

↑ "given" or "conditional on"

We will continue our discussion of conditional probability.

Multiplication Property

If $\mathbb{P}(A_1) > 0$:

rearrangement
of def'n

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_2 | A_1) \mathbb{P}(A_1).$$

COROLLARY

If $\mathbb{P}(A_1), \mathbb{P}(A_1 \cap A_2), \dots, \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_{n-1}) > 0$, then

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) &= \mathbb{P}(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}) \\ &\quad \times \mathbb{P}(A_{n-1} | A_1 \cap A_2 \cap \dots \cap A_{n-2}) \\ &\quad \times \dots \times \\ &\quad \times \mathbb{P}(A_3 | A_1 \cap A_2) \times \mathbb{P}(A_2 | A_1) \times \mathbb{P}(A_1) \end{aligned}$$

Proof of multiplication property (Informal-ish)

Consider $n = 4$

$$P(A_1) P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot P(A_4 | A_1 \cap A_2 \cap A_3)$$

$$= \frac{\cancel{P(A_1)} \cancel{P(A_1 \cap A_2)}}{\cancel{P(A_1)}} \cdot \frac{\cancel{P(A_1 \cap A_2 \cap A_3)}}{\cancel{P(A_1 \cap A_2)}} \cdot \frac{P(A_1 \cap A_2 \cap A_3 \cap A_4)}{\cancel{P(A_1 \cap A_2 \cap A_3)}}$$

$$= P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

Urns and Balls

EXERCISE: URNS AND BALLS

- An urn has 10 red balls and 40 black balls.
- Three balls are randomly drawn without replacement.

Calculate the probability that:

- a. The 3rd ball is red given that the 1st is red and the 2nd is black. $P(R_3 | R_1, B_2)$
- b. The first drawn ball is red, the 2nd is black and the 3rd is red. $P(R_1 \cap B_2 \cap R_3)$

$$R_i = \{ i^{\text{th}} \text{ ball is red} \}$$
$$B_i = \{ i^{\text{th}} \text{ ball is black} \}$$

Urns and Balls

10 red

40 black

$$a) P(R_3 | R_1, B_2)$$

$$= \frac{9}{(9+39)} = \frac{9}{48}$$

9 red



→ If first ball is red → 9 Red remaining
→ 2nd black → 39 black remaining

OR

$$P(R_3 | R_1, B_2) = \frac{P(R_1 \cap B_2 \cap R_3)}{P(R_1 \cap B_2)} = \frac{\frac{10}{50} \cdot \frac{40}{49} \cdot \frac{9}{48}}{\frac{10}{50} \cdot \frac{40}{49}} = \frac{9}{48}$$

$$b) \frac{10}{50} \cdot \frac{40}{49} \cdot \frac{9}{48} \approx 0.031$$

The “Total Probability” Formula

DEFINITION

We say that B_1, \dots, B_n is a **partition** of Ω if

1. They are disjoint

$$B_i \cap B_j = \emptyset \quad \text{for } i \neq j,$$

2. They cover the whole sample space: $\bigcup_{i=1}^n B_i = \Omega$

A simple partition is any event A and its complement A^c .

The “Total Probability” Formula

THEOREM

If B_1, \dots, B_n is a **partition** of Ω , then, for any $A \subseteq \Omega$,

$$\mathbb{P}(A) = \sum_{i=1}^n \mathbb{P}(A | B_i) \mathbb{P}(B_i).$$

↳ more general

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

* finite?

Proof of Total Probability

PROOF

- $A = A \cap \Omega = A \cap \left(\bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n (A \cap B_i)$
- The events $(A \cap B_i)$ are disjoint.
- Therefore, by Axiom 3, we have

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\left(\bigcup_{i=1}^n A \cap B_i\right) \\ &= \sum_{i=1}^n \mathbb{P}(A \cap B_i) \\ &= \sum_{i=1}^n \mathbb{P}(A | B_i) \mathbb{P}(B_i). \end{aligned}$$

Flu Test

- Suppose that every patient who visits the ER is given a flu test.
- Suppose that 30% of patients have flu. $P(A) = 0.3 \Rightarrow P(A^c) = 1 - 0.3 = 0.7$
- A patient with flu tests positive 90% of the time. $P(B|A) = 0.9$
- A patient without flu tests negative 80% of the time. $P(B^c|A^c) = 0.80$

EXERCISE: FLU TEST

If a new patient walks into the ER, what is the probability that they test positive for the flu? $P(\quad)?$

$A = \{ \text{patient has flu} \}$

$B = \{ \text{patient tests positive} \}$

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= 0.9 \cdot 0.3 + (1 - P(B^c|A^c))P(A^c) \\ &= 0.9 \cdot 0.3 + (1 - 0.8)(0.7) \\ &= 0.41 \end{aligned}$$

Flu Test

Bayes' Theorem

- Sometimes we have information about $\mathbb{P}(A | B)$ but require $\mathbb{P}(B | A)$.
- Bayes' Theorem allows us to relate these conditional probabilities.

THEOREM

Bayes' Theorem

Let A and B be arbitrary sets with $\mathbb{P}(A) > 0$. we have

$$\mathbb{P}(B | A) = \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\mathbb{P}(A)}$$

You can also consider B_1, B_2, \dots, B_n is a partition of Ω , then for each $i = 1, \dots, n$, so that:

$$\mathbb{P}(B | A) = \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A | B_j) \mathbb{P}(B_j)}$$

Proof of Bayes Formula

 PROOF

$$\begin{aligned}\mathbb{P}(B | A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} && \text{(Definition of conditional prob)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\mathbb{P}(A)} && \text{(Multiplication Rule)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A | B_j) \mathbb{P}(B_j)} && \text{(Rule of Total Prob)}\end{aligned}$$

Bayes' Theorem

In general, $\mathbb{P}(A \mid B) \neq \mathbb{P}(B \mid A)$. The assumption that these two probabilities are equivalent is referred to as the “conditional probability fallacy” or “confusion of the inverse”

- In some court systems, presenting evidence as a conditional probability has been banned due to the frequent confusion of the inverse
- For example:
 $\mathbb{P}(\text{DNA found at the crime scene} \mid \text{Guilty}) \neq \mathbb{P}(\text{Guilty} \mid \text{DNA found at the crime scene})$

Flu Prevalence

- Suppose that every patient who visits the ER is given a flu test.
- Suppose that 30% of tests are positive. $P(B) = 0.30$
- A patient with flu tests positive 90% of the time. $P(B|A) = 0.90$
- A patient without flu tests negative 80% of the time. $P(B^c|A^c) = 0.80$

EXERCISE: FLU TEST #2

Suppose you bring a friend to the ER.

- a. What is the probability that your friend has the flu if they test positive? $P(A|B)$
- b. What is the probability that your friend has the flu if they test negative? $P(A|B^c)$

$A = \{ \text{patient has flu} \}$

$B = \{ \text{patient tests positive} \}$

Flu Prevalence

$A = \{ \text{patient has flu} \}$

$B = \{ \text{patient tests positive} \}$

$$P(B) = 0.30$$

$$P(B|A) = 0.90$$

$$P(B^c|A^c) = 0.80$$

$$a) P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes

need $P(A) \dots$

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) \quad \text{L.T.P}$$

$$P(B) = P(B|A)P(A) + (1 - P(B^c|A^c))(1 - P(A))$$

$$0.3 = 0.9P(A) + (1 - 0.80)(1 - P(A)) \quad \text{comp rule x2}$$

$$0.3 = 0.9P(A) + 0.2 - 0.2P(A)$$

$$0.1 = 0.7P(A)$$

$$\frac{1}{7} = P(A)$$



Flu Prevalence

$$\begin{aligned}P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\&= \frac{0.9(1/7)}{0.3} \\&= 3/7 \approx 0.429\end{aligned}$$

$$b) P(A|B^c) = \frac{P(B^c|A)P(A)}{P(B^c)}$$



$$= \frac{(1 - P(B|A))P(A)}{(1 - P(B))}$$

comp
rule

$$= \frac{(1 - 0.9)(1/7)}{1 - 0.3}$$

$$= 0.020$$

The Monty Hall Problem

- You are a contestant on a game show. In front of you are three doors.
- Behind two doors are goat. 
- Behind one door is a car 

You select a door, the host then opens one of the 2 remaining doors, revealing a goat . The host asks

Would you like to switch to the remaining closed door?

What would you do? Discuss with your peers.

The Monty Hall Problem

EXERCISE: MONTY HALL

Show that the probability of winning the car if you switch doors is $2/3$.

The trick: correctly defining the sample space.

$$Y = \{ \text{door you choose} \} = \{ 1, 2, 3 \}$$

$$R = \{ \text{door with the car} \} = \{ 1, 2, 3 \}$$

$$H = \{ \text{door host opens} \} = \{ 1, 2, 3 \}$$

$$\Omega = \{ (Y=y, R=r, H=h) : y, r, h \in \{1, 2, 3\}, y \neq h, r \neq h \}$$

↳ not 27 combinations as host will not open your door or the prize door.

The Monty Hall Problem

Assume WLOG, $Y=1$: $\Omega = \{(y, r, h)\} = \{(1, 1, 2), (1, 1, 3), (1, 2, 3), (1, 3, 2)\}$

"Win" = $\{Y=R\}$

$$\begin{aligned} P(\text{Win}) &= P(\text{Win} | H=1)P(H=1) + P(\text{Win} | H=2)P(H=2) + P(\text{Win} | H=3)P(H=3) \\ &= 0 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{3} \leftarrow P(\text{Win}) \text{ if you don't switch doors} \end{aligned}$$

$$P(\text{lose}) = 1 - P(\text{Win}) = \frac{2}{3} \leftarrow P(\text{Win}) \text{ if you did switch!!!}$$

2 Independence

Independence

📖 DEFINITION

Independence:

We say that events A and B are independent if

$$\ast \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

📐 THEOREM

If $\mathbb{P}(B) > 0$ and A and B are independent events, then:

$$\ast \mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

- Knowledge about B occurring does not change the probability of A and vice versa.

Independence

DEFINITION

We say that an event A is non-trivial if $0 < P(A) < 1$.

THEOREM

If A and B are non-trivial events. Then,

- a. If $A \cap B = \emptyset$ then A and B are not independent
- b. If $A \subset B$ then A and B are not independent.

Independence

 PROOF

$$\text{a. } \mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0 \neq \mathbb{P}(A)$$

$\therefore A, B$ not independent

$$\text{b. } \mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \neq \mathbb{P}(A)$$

$\therefore A, B$ not independent

Independence and Complements

EXERCISE: INDEPENDENCE AND COMPLEMENTS

Show the following:

a. If A and B are independent then so are A^c and B .

b. If A and B are independent then so are A and B^c .

(Try at home)

*c. (~~Try at home!~~) If A and B are independent then so are A^c and B^c

↳ start with $P(A^c \cap B^c)$

↳ De Morgan's i

$$\begin{aligned} \text{a) } P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \quad \text{since independent} \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c) \end{aligned}$$

Independence and Complements

$$\begin{aligned} a) P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \quad \text{since } A, B \text{ independent} \\ &= P(B)(1 - P(A)) \\ &= P(B)P(A^c) \end{aligned}$$

$$\begin{aligned} c) P(A^c \cap B^c) &= P[(A \cup B)^c] \quad \text{De Morgan} \\ &= 1 - P(A \cup B) \quad \text{comp rule} \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [P(A) + P(B) - P(A)P(B)] \quad A, B \text{ independent} \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= \underline{1 - P(A)} - P(B)(\underline{1 - P(A)}) = (1 - P(B))(1 - P(A)) \\ &= P(B^c)P(A^c) \end{aligned}$$

More than 2 Independent Events

📖 DEFINITION

We say that the events A_1, A_2, \dots are independent if, for any finite collection $K = \{(i_1, \dots, i_k)\}$,

$$\mathbb{P} \left(\bigcap_{i \in K} A_i \right) = \prod_{i \in K} \mathbb{P}(A_i).$$

For example, if $n = 3$, then, A_1, A_2 , and A_3 are **independent if and only if** all of the following hold:

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1) \mathbb{P}(A_2), \\ \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1) \mathbb{P}(A_3), \\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2) \mathbb{P}(A_3), \\ \mathbb{P}(A_1 \cap A_2 \cap A_3) &= \mathbb{P}(A_1) \mathbb{P}(A_2) \mathbb{P}(A_3). \end{aligned} \quad \left. \vphantom{\begin{aligned} \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1) \mathbb{P}(A_2), \\ \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1) \mathbb{P}(A_3), \\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2) \mathbb{P}(A_3), \end{aligned}} \right\} \text{Pairwise}$$

Coin Flipping

We flip a fair coin twice. Define the following three events:

1. $A = \{\text{first flip is H}\}$.
2. $B = \{\text{second flip is H}\}$.
3. $C = \{\text{flips show the same result}\}$.

EXERCISE: PAIRWISE INDEPENDENCE

Show that A, B, C are pairwise independent, but not independent.

Coin Flipping

$$\Omega = \{HH, TT, HT, TH\}, \quad P(\omega) = 1/4 \quad \forall \omega \in \Omega$$

$$P(A) = P(\text{first flip H}) = 1/2$$

$$P(B) = P(\text{second flip H}) = 1/2$$

$$P(C) = P(\text{flips same}) = 1/2$$

Pairwise:

$$P(A \cap B) = 1/4 \quad P(A) \cdot P(B) = 1/2 \cdot 1/2 = 1/4$$

$$P(A \cap C) = 1/4 \quad P(A) \cdot P(C) = 1/4$$

$$P(B \cap C) = 1/4 \quad P(B) \cdot P(C) = 1/4$$

Yes, pairwise independent

$$P(A \cap B \cap C) = P(\{HH\}) = \underline{1/4}$$

$$P(A)P(B)P(C) = 1/2 \cdot 1/2 \cdot 1/2 = \underline{1/8}$$

\therefore Not independent

To do:

- Prove the last theorem in your own time (good practice!)
- Read **Chapters 1.5.2, 2.1, 2.2** [↗](#) before Wednesday's class (**SERIOUSLY**)
- Submit Assignment 1 by Wednesday May 20th, 11:59pm