

Lecture 5

Conditional Probability (Part 2)

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Last modified — 22 Sep 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Apply Bayes theorem
- Determine whether events are independence based on conditional probabilities
- Prove results where events are independent

Bayes' Theorem

- Sometimes we have information about $\mathbb{P}(A \mid B)$ but require $\mathbb{P}(B \mid A)$.
- Bayes' Theorem allows us to relate these conditional probabilities.

THEOREM

Bayes' Theorem

Let A and B be arbitrary sets with $\mathbb{P}(A) > 0$. we have

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\mathbb{P}(A)}$$

You can also consider $B_1, B_2, \dots, B_n$ is a partition of Ω , then for each $i = 1, \dots, n$, so that:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A \mid B_j) \mathbb{P}(B_j)}$$

Proof of Bayes Formula

 PROOF

$$\begin{aligned}\mathbb{P}(B | A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} && \text{(Definition of conditional prob)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\mathbb{P}(A)} && \text{(Multiplication Rule)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A | B_j) \mathbb{P}(B_j)} && \text{(Rule of Total Prob)}\end{aligned}$$

Note for Graue:
*change to C_j 's in future

Bayes' Theorem

In general, $\mathbb{P}(A | B) \neq \mathbb{P}(B | A)$. The assumption that these two probabilities are equivalent is referred to as the “conditional probability fallacy” or “confusion of the inverse”

- In some court systems, presenting evidence as a conditional probability has been banned due to the frequent confusion of the inverse

- For example:

$$\mathbb{P}(\text{DNA found at the crime scene} | \text{Guilty}) \neq \mathbb{P}(\text{Guilty} | \text{DNA found at the crime scene})$$

When are they equal?

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

when $\frac{P(B)}{P(A)} = 1$ OR $P(B) = P(A)$

Flu Prevalence

- Suppose that every patient who visits the ER is given a flu test.
- Suppose that 30% of tests are positive. $P(B) = 0.3$
- A patient with flu tests positive 90% of the time. $P(B|A) = 0.9$
- A patient without flu tests negative 80% of the time. $P(B^c|A^c) = 0.8$

EXERCISE: FLU TEST #2

Suppose you bring a friend to the ER.

- a. What is the probability that your friend has the flu if they test positive? $P(A|B)$
- b. What is the probability that your friend has the flu if they test negative? $P(A|B^c)$

A: Patient has flu

B: test positive

Flu Prevalence

$$a) P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad (\text{Bayes's})$$

Need $P(A)$...

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

L.T.P

$$P(B) = P(B|A)P(A) + (1 - P(B^c|A^c))(1 - P(A)) \quad \text{comp. rule}$$

$$0.3 = 0.9 P(A) + (1 - 0.8)(1 - P(A))$$

$$0.3 = 0.9 P(A) + 0.2 - 0.2 P(A)$$

$$0.1 = 0.7 P(A)$$

$$\hookrightarrow P(A) = 1/7$$

$\therefore$

$$P(A|B) = \frac{0.9 \cdot (1/7)}{0.3}$$

$$= \boxed{0.429}$$

Flu Prevalence

$$\begin{aligned} b) \quad P(A|B^c) &= \frac{P(B^c|A)P(A)}{P(B^c)} \\ &= \frac{[1 - P(B|A)]P(A)}{1 - P(B)} \\ &= \frac{(1 - 0.9)(1/7)}{1 - 0.3} \\ &= \boxed{0.02} \end{aligned}$$

$$P(B) = 0.3$$

$$P(B|A) = 0.9$$

$$P(B^c|A^c) = 0.8$$

1 Independence

Independence

DEFINITION

Independence:

We say that events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B). \quad (\text{from lec 2})$$

THEOREM

If $\mathbb{P}(B) > 0$ and A and B are independent events, then:

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

$$P(A|B) = P(A) \quad \text{when } A, B \text{ are independent.}$$

- Knowledge about B occurring does not change the probability of A and vice versa.

Independence

DEFINITION

We say that an event A is non-trivial if $0 < P(A) < 1$.

THEOREM

If A and B are non-trivial events. Then,

- a. If $A \cap B = \emptyset$ then A and B are not independent
- b. If $A \subset B$ then A and B are not independent.

A : wear a brown sweater
 B : wear " pink "

$$A \cap B = \emptyset$$

A is not independent of B .

Independence

Proof:

a) let $A \cap B = \emptyset$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0 \neq P(A) \text{ since } 0 < P(A) < 1 \text{ (non-trivial)}$$

$\therefore$ not independent.

b) Let $A \subseteq B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \neq P(A)$$

$\therefore$ not independent.

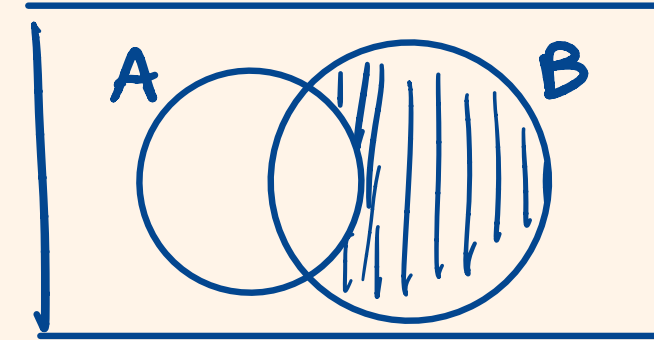
since $P(B) \neq 1$
(both are non-trivial!)

Independence and Complements

EXERCISE: INDEPENDENCE AND COMPLEMENTS

Show the following:

- a. If A and B are independent then so are A^c and B .
- b. If A and B are independent then so are A and B^c .
- c. If A and B are independent then so are A^c and B^c .



$$\begin{aligned} \text{a) } \underline{P(A^c \cap B)} &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \text{ since independent.} \\ &= P(B)(1 - P(A)) \\ &= \underline{P(B)P(A^c)} \quad \checkmark \end{aligned}$$

Independence and Complements

$$\begin{aligned} c) P(A^c \cap B^c) &= P[(A \cup B)^c] && \text{De Morgan's Law} \\ &= 1 - P(A \cup B) && \text{comp rule} \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [P(A) + P(B) - P(A)P(B)] && \text{since } A, B \text{ indep.} \\ &= (1 - P(A)) - \underline{P(B) + P(A)P(B)} \\ &= 1(1 - P(A)) - \underline{P(B)(1 - P(A))} && \text{factoring out } P(B) \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c) \end{aligned}$$



b) Make sure you can do this!!!

More than 2 Independent Events

DEFINITION

We say that the events $A_1, A_2, \dots$ are independent if, for any finite collection $K = \{i_1, \dots, i_k\}$,

$$\mathbb{P} \left(\bigcap_{i \in K} A_i \right) = \prod_{i \in K} \mathbb{P}(A_i).$$

$$P(A \cap B) = P(A)P(B)$$

For example, if $n = 3$, then, A_1, A_2 , and A_3 are **independent if and only if all of the following hold:**

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \mathbb{P}(A_2),$$

$$\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1) \mathbb{P}(A_3),$$

$$\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2) \mathbb{P}(A_3),$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \mathbb{P}(A_2) \mathbb{P}(A_3).$$

Coin Flipping

We flip a fair coin twice. Define the following three events:

1. $A = \{\text{first flip is H}\}.$
2. $B = \{\text{second flip is H}\}.$
3. $C = \{\text{flips show the same result}\}.$

EXERCISE: PAIRWISE INDEPENDENCE

Show that A, B, C are pairwise independent, but not independent.

Coin Flipping

$$\Omega = \{HH, TT, HT, TH\} \quad P(\omega) = 1/4 \quad \omega \in \Omega$$

$$P(A) = P(\text{first flip } H) = 2/4 = 1/2$$

$$P(B) = P(\text{second } H) = 2/4 = 1/2$$

$$P(C) = P(\text{flip same}) = 2/4 = 1/2$$

Check pairwise independent:

$$P(A \cap B) = P(\{HH\}) = 1/4 \quad P(A)P(B) = 1/2 \cdot 1/2 = 1/4$$

$$P(A \cap C) = 1/4$$

$$P(A)P(C) = 1/2 \cdot 1/2 = 1/4$$

$$P(B \cap C) = 1/4$$

$$P(B)P(C) = 1/4$$

Yes, pairwise ind ✓

$$\begin{aligned} P(A \cap B \cap C) &= P(\{HH\}) = 1/4 \end{aligned}$$

$$\begin{aligned} P(A)P(B)P(C) &= 1/2 \cdot 1/2 \cdot 1/2 \\ &= 1/8 \end{aligned}$$

$$P(A \cap B \cap C) \neq P(A)P(B)P(C)$$

NOT IND

Checkpoint 1

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing calculator
- Writing materials

Phones and laptops put away, please! You can have 15 minutes.

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Checkpoint 2 (on Section 1.5: Conditional Probability and Independence) is next Friday!

Same rules as Checkpoint 1!