

Lecture 5

Conditional Probability (Part 2)

Grace Tompkins

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Apply Bayes theorem
- Determine whether events are independence based on conditional probabilities
- Prove results where events are independent

Bayes' Theorem

- Sometimes we have information about $\mathbb{P}(A \mid B)$ but require $\mathbb{P}(B \mid A)$.
- Bayes' Theorem allows us to relate these conditional probabilities.

THEOREM

Bayes' Theorem

Let A and B be arbitrary sets with $\mathbb{P}(A) > 0$. we have

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\mathbb{P}(A)}$$

You can also consider $B_1, B_2, \dots, B_n$ is a partition of Ω , then for each $i = 1, \dots, n$, so that:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A \mid B_j) \mathbb{P}(B_j)}$$

Proof of Bayes Formula

 PROOF

$$\begin{aligned}\mathbb{P}(B | A) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} && \text{(Definition of conditional prob)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\mathbb{P}(A)} && \text{(Multiplication Rule)} \\ &= \frac{\mathbb{P}(A | B) \mathbb{P}(B)}{\sum_{j=1}^n \mathbb{P}(A | B_j) \mathbb{P}(B_j)} && \text{(Rule of Total Prob)}\end{aligned}$$

Bayes' Theorem

In general, $\mathbb{P}(A \mid B) \neq \mathbb{P}(B \mid A)$. The assumption that these two probabilities are equivalent is referred to as the “conditional probability fallacy” or “confusion of the inverse”

- In some court systems, presenting evidence as a conditional probability has been banned due to the frequent confusion of the inverse
- For example:
 $\mathbb{P}(\text{DNA found at the crime scene} \mid \text{Guilty}) \neq \mathbb{P}(\text{Guilty} \mid \text{DNA found at the crime scene})$

Flu Prevalence

- Suppose that every patient who visits the ER is given a flu test.
- Suppose that 30% of tests are positive.
- A patient with flu tests positive 90% of the time.
- A patient without flu tests negative 80% of the time.

EXERCISE: FLU TEST #2

Suppose you bring a friend to the ER.

- a. What is the probability that your friend has the flu if they test positive?
- b. What is the probability that your friend has the flu if they test negative?

Flu Prevalence

Flu Prevalence

1 Independence

Independence

DEFINITION

Independence:

We say that events A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B).$$

THEOREM

If $\mathbb{P}(B) > 0$ and A and B are independent events, then:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A) \mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$$

- Knowledge about B occurring does not change the probability of A and vice versa.

Independence

DEFINITION

We say that an event A is non-trivial if $0 < P(A) < 1$.

THEOREM

If A and B are non-trivial events. Then,

- a. If $A \cap B = \emptyset$ then A and B are not independent
- b. If $A \subset B$ then A and B are not independent.

Independence

Proof:

Independence and Complements

EXERCISE: INDEPENDENCE AND COMPLEMENTS

Show the following:

- a. If A and B are independent then so are A^c and B .
- b. If A and B are independent then so are A and B^c .
- c. If A and B are independent then so are A^c and B^c .

Independence and Complements

More than 2 Independent Events

DEFINITION

We say that the events $A_1, A_2, \dots$ are independent if, for any finite collection $K = \{i_1, \dots, i_k\}$,

$$\mathbb{P} \left(\bigcap_{i \in K} A_i \right) = \prod_{i \in K} \mathbb{P}(A_i).$$

For example, if $n = 3$, then, A_1, A_2 , and A_3 are **independent if and only if all of the following hold**:

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2) &= \mathbb{P}(A_1) \mathbb{P}(A_2), \\ \mathbb{P}(A_1 \cap A_3) &= \mathbb{P}(A_1) \mathbb{P}(A_3), \\ \mathbb{P}(A_2 \cap A_3) &= \mathbb{P}(A_2) \mathbb{P}(A_3), \\ \mathbb{P}(A_1 \cap A_2 \cap A_3) &= \mathbb{P}(A_1) \mathbb{P}(A_2) \mathbb{P}(A_3). \end{aligned}$$

Coin Flipping

We flip a fair coin twice. Define the following three events:

1. $A = \{\text{first flip is H}\}.$
2. $B = \{\text{second flip is H}\}.$
3. $C = \{\text{flips show the same result}\}.$

EXERCISE: PAIRWISE INDEPENDENCE

Show that A, B, C are pairwise independent, but not independent.

Coin Flipping

Checkpoint 1

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing calculator
- Writing materials

Phones and laptops put away, please! You can have 15 minutes.

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Checkpoint 2 (on Section 1.4: Conditional Probability and Independence) is next Friday!

Same rules as Checkpoint 1!