

Lecture 5

~~Independence~~, Random Variables, and Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a random variable, and a distribution
- Identify and define appropriate random variables from word problems
- Write out distributions using indicator functions

1 Random Variables

Review

- So far, we have discussed how probability measures (functions) operate on events (subsets) of the sample space.
- For any event $A \subseteq \Omega$, $\mathbb{P}(A)$ is a number between 0 and 1.
- But events are not always the most natural way to talk about possible outcomes of an experiment.
- It is often easier to describe an event $A \subseteq \Omega$ with a single number.

EXAMPLE

- Recall our experiment “roll a die until you see 6”.
- There are many sequences of rolls (elements of Ω) in the event “first 6 on roll 12”: E_{12} .
- What if we summarize these with “12”?

Random Variables

- A random variable is a way of summarizing events mathematically.
- We use capital letters (usually, near the end of alphabet) to denote random variables (X, Y, Z , etc.)

DEFINITION

Random variable: a **function** from the sample space to a subset of the real numbers.

Therefore, a the random variable called X is any function

$$X : \Omega \rightarrow \mathbb{R}.$$

That is, for each $\omega \in \Omega$,

$$X(\omega) \in \mathbb{R}.$$

Random Variables

💡 EXAMPLE

Suppose we toss a coin 3 times.

- Sample space: $\Omega = \left\{ (x_1, x_2, x_3) : x_i \in \{H, T\} \right\}$

Let $X(\omega)$ = number of heads in ω .

- If $\omega = (HHH)$, $X(\omega) = 3$
- If $\omega = (HHT)$, (HTH) , or (THH) , $X(\omega) = 2$
- ... and so on

We can summarize this as:

$$X(\omega) =$$

0

$$\omega = TTT$$

1

$$\omega = HTT, THT, TTH$$

2

$$\omega = HHT, HTH, THH$$

3

$$\omega = HHH$$

Random Variables

EXERCISE: WEATHER

Suppose it can either be raining, snowy, or sunny.

Let $X = 3$ if its raining, $X = 6$ if it snows, and $X = -2.7$ if it's sunny.

Is X a random variable?

Yes! $X(\omega)$ is defined for all $\omega \in \Omega$, and
maps $\Omega \rightarrow \mathbb{R}$

Random Variables

💡 EXAMPLE

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $X(\omega) = \omega$ and $Y(\omega) = \omega^3 + 2$.

Compute $Z(\omega)$ for all $\omega \in \Omega$ where $Z = XY$.

$$Z(\omega) = \omega(\omega^3 + 2) =$$

$$(1)(1^3 + 2) = 3$$

$$(2)(2^3 + 2) = 20$$

$$(3)(3^3 + 2) = 87$$

$$264$$

$$635$$

$$1308$$

$$\omega = 1$$

$$\omega = 2$$

$$\omega = 3$$

$$\omega = 4$$

$$\omega = 5$$

$$\omega = 6$$

Canucks Hockey

- Suppose the Canucks play 2 games next week.
- Each game can be won (W), lost (L), or lost in overtime (O).
- The Canucks get points for wins (2), overtime losses (1), and losses (0).

EXERCISE:

- Write out the sample space Ω .
- Define $Z = \#$ games played. Is Z a random variable? If so, enumerate how it operates on each $\omega \in \Omega$
- Define $X = \#$ points. Enumerate how it operates on each $\omega \in \Omega$.
- Which outcomes are in the event $A = \{X \geq 3\}$?

$$a) \Omega = \{WW, LL, OO, WL, LW, WO, OW, LO, OL\}$$

Canucks Hockey

W = 2 pts, O = 1 pt, L = 0 pt

$$a) \Omega = \{WW, LL, OO, WL, LW, WO, OW, LO, OL\}$$

b) $Z(\omega) = 2 \quad \forall \omega \in \Omega$. $Z: \Omega \rightarrow \mathbb{R}$ so yes it a RV
random variable

$$c) X(\omega) = \begin{cases} 0 & \omega = LL \\ 1 & \omega = OL, LO \\ 2 & \omega = OO, WL, LW \\ 3 & \omega = OW, WO \\ 4 & \omega = WW \end{cases}$$

$$d) A = \{X \geq 3\} = \{OW, WO, WW\}$$

Random Variables and Events

- Random variables are naturally used to describe events of interest.
- For example $\{X > 3\}$ or $\{Y \leq 2\}$
- Formally, this notation means:

$$\{X > 3\} = \{\omega \in \Omega : X(\omega) > 3\},$$

and

$$\{Y \leq 2\} = \{\omega \in \Omega : Y(\omega) \leq 2\}.$$

In other words, $\{X > 3\}$ and $\{Y \leq 2\}$ are events.

The Indicator Function

$$I_A \quad \mathbb{I}_A \quad \mathbb{1}_A$$

The indicator function is a useful mathematical shorthand.

It is also a random variable: it is a function from $\Omega \rightarrow \mathbb{R}$.

DEFINITION

The indicator of the event A is the random variable given by

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{else.} \end{cases}$$

The Indicator Function

EXAMPLE

Let

$$A = (1, 2, 3, 4, 5, 6)$$

$I_A(5) = 1$ because 5 is in the set A.

$I_A(9) = 0$ because 9 is not in the set A.

The Indicator Function

💡 EXAMPLE

Consider a 6-sided fair die, where $\Omega = \{1, 2, 3, 4, 5, 6\}$

What is $Y = X + I_{\{6\}}$? $X(\omega) = \omega$

$$\begin{aligned} Y(\omega) &= X(\omega) + I_{\{6\}}(\omega) = \\ &= \omega + I_{\{6\}}(\omega) = \begin{cases} \omega, & \omega \leq 5 \\ b+1 = 7, & \omega = 6 \end{cases} \\ &\quad \downarrow \\ &= 1 \text{ if } \omega = 6 \\ &\quad 0 \text{ otherwise} \end{aligned}$$

The Indicator Function

EXERCISE: PRODUCT OF INDICATORS

Let A and B be events, and let $X = I_A \times I_B$. Is X an indicator function? If so, of what event?

$$X(\omega) = \begin{cases} 1 & , \quad I_A(\omega) = 1 \text{ and } I_B(\omega) = 1 \\ 0 & \quad \text{o/w} \end{cases}$$

$$I_A(\omega) = 1 \rightarrow \omega \in A$$

$$I_B(\omega) = 1 \rightarrow \omega \in B$$

$$X = I_{(A \cap B)}$$

2 Distributions

Distributions

- A probability associates each $\omega \in \Omega$ with a number between 0 and 1 and satisfies the Probability axioms.
- Random variables also operate on Ω . Taking these together **induces** a probability on \mathbb{R} .

💡 EXAMPLE

Suppose we flip a fair coin with $\Omega = \{H, T\}$, and we know that $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$.

Then, considering the RV I_H , we see $\mathbb{P}(I_H = 1) = \mathbb{P}(\{\omega : I_H(\omega) = 1\}) = \mathbb{P}(H)$

Distributions

DEFINITION

Distribution: If X is a random variable, then the **distribution** of X is the collection of probabilities $\mathbb{P}(X \in B)$ for all subsets B of \mathbb{R} .

Consider $\Omega = \{H, T\}$, and $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$

- This defines \mathbb{P} on all subsets $2^\Omega = \{\emptyset, \{T\}, \{H\}, \Omega\}$.
- But considering a random variable X , it isn't really enough to know its probability only on its range.
- We need to know more than that: we need to know, for **every** $B \subset \mathbb{R}$, what is $\mathbb{P}(X \in B) = \mathbb{P}(\{s \in \Omega : X(s) \in B\})$.
- **To fully specify the distribution**, we need to do this for every (nice) subset B . The collection of these subsets is denoted \mathcal{B} .

The mathematical details are at least at the level of Math 420.

Distribution Example

💡 EXAMPLE

Consider shuffling a deck of 50 Pokemon cards where 20 are grass-type (G), 13 are fighting-type (F), 7 are water-type (W), and 10 are electric-type (E). Professor Grace really likes the grass and water types.

Consider the random variable Z , which indicates if Grace will *really like* the card chosen. Compute $\mathbb{P}(Z = z)$ for any $z \in \mathbb{R}$

$$P(W) = \begin{cases} 0.40 & W=G \\ 0.26 & W=F \\ 0.14 & W=W \\ 0.20 & W=E \end{cases}$$

$$Z = I(\text{Grace likes } \overset{\text{grass or water}}{\text{card}}) \\ = I_{G \cup W}$$

$$R_Z = \{0, 1\} \\ \hookrightarrow \text{possible values for } Z$$

Distribution Example

$$\begin{aligned}P(Z=0) &= P(\{\omega: Z(\omega)=0\}) \\ &= P(\{E \cup F\}) \quad \leftarrow \text{not water or grass type} \\ &= P(E) + P(F) \\ &= 0.20 + 0.26 \\ &= \underline{0.46}\end{aligned}$$

$$\begin{aligned}P(Z=1) &= P(\{\omega: Z(\omega)=1\}) \\ &= P(W \cup G) \\ &= P(W) + P(G) \\ &= 0.14 + 0.40 \\ &= \underline{0.54}\end{aligned}$$

prob. sum to 1!

$$P(Z=z) = \begin{cases} 0.46 & z=0 \\ 0.54 & z=1 \\ \underline{0} & \text{otherwise} \end{cases}$$

↑
must include
all possible $z \in \mathbb{R}$

More Hockey

- Suppose the Canucks play 2 games next week. Each game can be won (W), lost (L), or lost in overtime (O). The Canucks get points for wins (2), overtime losses (1), and losses (0).
- The Canucks are **not very good**. Suppose for each game (independently),

$$\mathbb{P}(\omega) = \begin{cases} 0.3 & \omega = W \\ 0.2 & \omega = O \\ 0.5 & \omega = L. \end{cases}$$

EXERCISE: SHOW THE CANUCKS ARE NOT VERY GOOD

- ★ a. Let X be the number of wins. Compute $\mathbb{P}(X = x)$ for all $x \in \mathbb{R}$. *Hint: Start by solving for $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, and $\mathbb{P}(X = 2)$*
- ★ b. Write a formula for $\mathbb{P}(X \in B)$ for any $B \subset \mathbb{R}$. *Hint: indicators!*
- c. Let Y be the number of points. Compute $\mathbb{P}(Y = y)$ for all $y \in \mathbb{R}$.

More Hockey

X = number of wins

$$R_x = \{0, 1, 2\}$$

$$\begin{aligned} P(X=0) &= P(\{\omega : X(\omega) = 0\}) = P(\{LL, OL, LO, OO\}) \\ &= P(W^c)P(W^c) \\ &= (1-0.3)(1-0.3) = 0.49 \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(\{\omega : X(\omega) = 1\}) = P(\{WL, LW, OW, WO\}) \\ &= (0.3)(0.5) + (0.5)(0.3) + (0.2)(0.3) + (0.3)(0.2) \\ &= 0.42 \end{aligned}$$

$$P(X=2) = P(\{WW\}) = 0.3 \cdot 0.3 = 0.09$$

More Hockey

$$a) P(X=x) = \begin{cases} 0.49 & x=0 \\ 0.42 & x=1 \\ 0.09 & x=2 \\ 0 & \text{o/w} \end{cases}$$

$$b) P(X \in B) = 0.49 I_B(0) + 0.42 I_B(1) + 0.09 I_B(2)$$

$$c) R_Y = \{0, 1, 2, 3, 4\}$$

$$P(Y=0) = P(\{\omega: Y(\omega)=0\}) = P(\{LL\}) = 0.5 \cdot 0.5 = 0.25$$

$$P(Y=1) = P(\{OL, LO\}) = (0.2)(0.5) + (0.5)(0.2) = 0.2$$

$$P(Y=2) = P(\{WL, LW, OO\}) = 2(0.3)(0.5) + (0.2)(0.2) = 0.34$$

⋮

More Hockey

$$P(Y=y) = \begin{cases} 0.25 & y=0 \\ 0.20 & y=1 \\ 0.34 & y=2 \\ 0.12 & y=3 \\ 0.09 & y=4 \\ 0 & \text{o/w} \end{cases}$$

"otherwise"

Midterm info:

- Your midterm will cover materials from Lectures 1 - 8
- DATE: in class on Tuesday June 2
- LENGTH: 1 hour and 50 minutes in length.
- You may bring in one (1) “cheat sheet”:
 - Must be HAND WRITTEN with pen/pencil on said sheet of paper (not typed, not photo copied, not printed, not written on an iPad)
 - Must be on 8.5 by 11 inch sheet of paper or smaller
 - You may write on both sides
 - No magnifying glasses or anything else silly
 - **I will confiscate cheatsheets that do not follow these rules** 🌹
 - I do not care what is written on it
- Exam is hand written on paper, bring something to write with
- You may bring a non-programmable, non-graphing calculator.

To do:

- Read **Chapters 2.3** [↗](#) before Friday's class
- Assignment 1 due TONIGHT 11:59pm. Submit on Gradescope.