Lecture 5

~~Independence~~, Random Variables, and Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a random variable, and a distribution
- Identify and define appropriate random variables from word problems
- Write out distributions using indicator functions

1 Random Variables

Review

- So far, we have discussed how probability measures (functions) operate on events (subsets) of the sample space.
- For any event $A \subseteq \Omega$, $\mathbb{P}(A)$ is a number between 0 and 1.
- But events are not always the most natural way to talk about possible outcomes of an experiment.
- It is often easier to describe an event $A \subseteq \Omega$ with a single number.

> 💡 EXAMPLE
>
> - Recall our experiment "roll a die until you see 6".
> - There are many sequences of rolls (elements of Ω) in the event "first 6 on roll 12": E_{12}.
> - What if we summarize these with "12"?

Random Variables

- A random variable is a way of summarizing events mathematically.
- We use capital letters (usually, near the end of alphabet) to denote random variables (X, Y, Z, etc.)

📖 DEFINITION

Random variable: a function from the sample space to a subset of the real numbers.

Therefore, a the random variable called X is any function

$$X \ : \ \Omega \ \to \mathbb{R}.$$

That is, for each $\omega \in \Omega$,

$$X(\omega) \in \mathbb{R}.$$

Random Variables

💡 EXAMPLE

Suppose we toss a coin 3 times.

- Sample space: $\Omega = \Big\{(x_1, x_2, x_3) \ : \ x_i \in \{H, T\}\Big\}$

Let $X(\omega) =$ number of heads in ω.

- If $\omega = (HHH)$, $X(\omega)$ = . 3
- If $\omega = (HHT), (HTH)$, or (THH), $X(\omega) =$. 2
- … and so on

We can summarize this as:

$$X(\omega) = \begin{cases} 0 & \omega = TTT \\ 1 & \omega = HTT, THT, TTH \\ 2 & \omega = HHT, HTH, THH \\ 3 & \omega = HHH \end{cases}$$

Random Variables

✏️ **EXERCISE: WEATHER**

Suppose it can either be raining, snowy, or sunny.

Let $X = 3$ if its raining, X = 6 if it snows, and X = -2.7 if it's sunny.

Is X a random variable?

Yes! $X(\omega)$ is defined for all $\omega \in \Omega$, and maps $\Omega \to \mathbb{R}$

Random Variables

💡 **EXAMPLE**

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $X(\omega) = \omega$ and $Y(\omega) = \omega^3 + 2$.

Compute $Z(\omega)$ for all $\omega \in \Omega$ where $Z = XY$.

$$Z(w) = w(w^3+2) = \begin{cases} (1)(1^3+2) = 3 & w=1 \\ (2)(2^3+2) = 12 & w=2 \\ (3)(3^3+2) = 33 & w=3 \\ 72 & w=4 \\ 135 & w=5 \\ 228 & w=6 \end{cases}$$

Canucks Hockey

- Suppose the Canucks play 2 games next week.
- Each game can be won (W), lost (L), or lost in overtime (O).
- The Canucks get points for wins (2), overtime losses (1), and losses (0).

✏️ **EXERCISE:**

a. Write out the sample space Ω.

b. Define $Z = \#\ \text{games played}$. Is Z a random variable? If so, enumerate how it operates on each $\omega \in \Omega$

c. Define $X = \#\ \text{points}$. Enumerate how it operates on each $\omega \in \Omega$.

d. Which outcomes are in the event $A = \{X \geq 3\}$?

a) $\Omega = \{WW, LL, OO, WL, LW, WO, OW, LO, OL\}$

Canucks Hockey

$W = 2$ pts, $O = 1$ pt, $L = 0$ pt

a) $\Omega = \{WW, LL, OO, WL, LW, WO, OW, LO, OL\}$

b) $Z(\omega) = 2 \quad \forall \omega \in \Omega$. $Z: \Omega \to \mathbb{R}$ so yes it a RV

random variable

c)
$$X(\omega) = \begin{cases} 0 & \omega = LL \\ 1 & \omega = OL, LO \\ 2 & \omega = OO, WL, LW \\ 3 & \omega = OW, WO \\ 4 & \omega = WW \end{cases}$$

d) $A = \{X \geq 3\} = \{OW, WO, WW\}$

Random Variables and Events

- Random variables are naturally used to describe events of interest.
- For example $\left\{X > 3\right\}$ or $\left\{Y \leq 2\right\}$
- Formally, this notation means:

$$\left\{X > 3\right\} = \left\{\omega \in \Omega \, : \, X(\omega) > 3\right\},$$

and

$$\left\{Y \leq 2\right\} = \left\{\omega \in \Omega \, : \, Y(\omega) \leq 2\right\}.$$

In other words, $\left\{X > 3\right\}$ and $\left\{Y \leq 2\right\}$ are events.

The Indicator Function

I_A $\mathbb{I}_A$ $\mathbb{1}_A$

The indicator function is a useful mathematical shorthand.

It is also a random variable: it is a function from $\Omega \to \mathbb{R}$.

📖 **DEFINITION**

The indicator of the event A is the random variable given by

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{else.} \end{cases}$$

The Indicator Function

💡 **EXAMPLE**

Let

$$A = (1, 2, 3, 4, 5, 6)$$

$I_A(5) = 1$ because 5 is in the set A.

$I_A(9) = 0$ because 9 is not in the set A.

The Indicator Function

> 💡 **EXAMPLE**
>
> Consider a 6-sided fair die, where $\Omega = \{1, 2, 3, 4, 5, 6\}$
>
> What is $Y = X + I_{\{6\}}$.? $\quad X(\omega) = \omega$

$$Y(\omega) = X(\omega) + I_{\{6\}}(\omega) =$$

$$= \omega + \underbrace{I_{\{6\}}(\omega)}_{\substack{=1 \text{ if } \omega = 6 \\ 0 \text{ otherwise}}} = \begin{cases} \omega & , \omega \leq 5 \\ 6+1=7 & , \omega = 6 \end{cases}$$

The Indicator Function

✏️ **EXERCISE: PRODUCT OF INDICATORS**

Let A and B be events, and let $X = I_A \times I_B$. Is X an indicator function? If so, of what event?

$$X(\omega) = \begin{cases} 1, & I_A(\omega) = 1 \text{ and } I_B(\omega) = 1 \\ 0 & \text{o/w} \end{cases}$$

$I_A(\omega) = 1 \rightarrow \omega \in A$

$I_B(\omega) = 1 \rightarrow \omega \in B$

$X = I_{(A \cap B)}$

2 Distributions

Distributions

- A probability associates each $\omega \in \Omega$ with a number between 0 and 1 and satisfies the Probability axioms.
- Random variables also operate on Ω. Taking these together induces a probability on $\mathbb{R}$.

💡 EXAMPLE

Suppose we flip a fair coin with $\Omega = \{H, T\}$, and we know that $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$.

Then, considering the RV I_H, we see $\mathbb{P}(I_H = 1) = \mathbb{P}(\{\omega : I_H(\omega) = 1\}) = \mathbb{P}(H)$

Distributions

📖 DEFINITION

Distribution: If X is a random variable, then the **distribution** of X is the collection of probabilities $\mathbb{P}(X \in B)$ for all subsets B of $\mathbb{R}$.

Consider $\Omega = \{H, T\}$, and $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$

- This defines $\mathbb{P}$ on all subsets $2^{\Omega} = \{\varnothing, \{T\}, \{H\}, \Omega\}$.
- But considering a random variable X, it isn't really enough to know it's probability only on it's range.
- We need to know more than that: we need to know, for every $B \subset \mathbb{R}$, what is $\mathbb{P}(X \in B) = \mathbb{P}(\{s \in \Omega : X(s) \in B\})$.
- To fully specify the distribution, we need to do this for every (nice) subset B. The collection of these subsets is denoted $\mathcal{B}$.

The mathematical details are at least at the level of Math 420.

Distribution Example

💡 EXAMPLE

Consider shuffling a deck of 50 Pokemon cards where 20 are grass-type (G), 13 are fighting-type (F), 7 are water-type (W), and 10 are electric-type (E). Professor Grace really likes the grass and water types.

Consider the random variable Z, which indicates if Grace will *really like* the card chosen. Compute $\mathbb{P}(Z = z)$ for any z. $\in \mathbb{R}$

$$P(w) = \begin{cases} 0.40 & w = G \\ 0.26 & w = F \\ 0.14 & w = W \\ 0.20 & w = E \end{cases}$$

grass or water

$$Z = I(\text{Grace likes card})$$
$$= I_{G \cup W}$$

$$R_Z = \{0, 1\}$$

↳ possible values for Z

Distribution Example

$$P(Z=0) = P(\{\omega : Z(\omega) = 0\})$$
$$= P(\{E \cup F\})$$ ← not water or grass type
$$= P(E) + P(F)$$
$$= 0.20 + 0.26$$
$$= 0.46$$

$$P(Z=1) = P(\{\omega : Z(\omega) = 1\})$$
$$= P(W \cup G)$$
$$= P(W) + P(G)$$
$$= 0.14 + 0.40$$
$$= 0.54$$

prob. sum to 1!

$$P(Z=z) = \begin{cases} 0.46 & z=0 \\ 0.54 & z=1 \\ 0 & \text{otherwise} \end{cases}$$

must include all possible $z \in \mathbb{R}$

More Hockey

- Suppose the Canucks play 2 games next week. Each game can be won (W), lost (L), or lost in overtime (O). The Canucks get points for wins (2), overtime losses (1), and losses (0).
- The Canucks are **not very good**. Suppose for each game (independently),

$$\mathbb{P}(\omega) = \begin{cases} 0.3 & \omega = W \\ 0.2 & \omega = O \\ 0.5 & \omega = L. \end{cases}$$

✏️ **EXERCISE: SHOW THE CANUCKS ARE NOT VERY GOOD**

a. Let X be the number of wins. Compute $\mathbb{P}(X = x)$ for all $x \in \mathbb{R}$. *Hint: Start by solving for $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, and $\mathbb{P}(X = 2)$*

b. Write a formula for $\mathbb{P}(X \in B)$ for any $B \subset \mathbb{R}$. *Hint: indicators!*

c. Let Y be the number of points. Compute $\mathbb{P}(Y = y)$ for all $y \in \mathbb{R}$.

More Hockey

$X =$ number of wins

$R_X = \{0, 1, 2\}$

$P(X=0) = P(\{w : X(w) = 0\}) = P(\{LL, OL, LO, OO\})$

$= P(W^c)P(W^c)$

$= (1-0.3)(1-0.3) = 0.49$

$P(X=1) = P(\{w : X(w) = 1\}) = P(\{WL, LW, OW, WO\})$

$= (0.3)(0.5) + (0.5)(0.3) + (0.2)(0.3) + (0.3)(0.2)$

$= 0.42$

$P(X=2) = P(\{WW\}) = 0.3 \cdot 0.3 = 0.09$

More Hockey

a)

$$P(X=x) = \begin{cases} 0.49 & x=0 \\ 0.42 & x=1 \\ 0.09 & x=2 \\ 0 & o/w \end{cases}$$

b) $P(X \in B) = 0.49 I_B(0) + 0.42 I_B(1) + 0.09 I_B(2)$

c) $R_y = \{0, 1, 2, 3, 4\}$

$P(Y=0) = P(\{w: Y(w)=0\}) = P(\{LL\}) = 0.5 \cdot 0.5 = 0.25$

$P(Y=1) = P(\{OL, LO\}) = (0.2)(0.5) + (0.5)(0.2) = 0.2$

$P(Y=2) = P(\{WL, LW, OO\}) = 2(0.3)(0.5) + (0.2)(0.2) = 0.34$

$\vdots$

More Hockey

$$P(Y=y) = \begin{cases} 0.25 & y=0 \\ 0.20 & y=1 \\ 0.34 & y=2 \\ 0.12 & y=3 \\ 0.09 & y=4 \\ 0 & o/w \end{cases}$$

"otherwise"

Midterm info:

- Your midterm will cover materials from Lectures 1 - 8
- DATE: in class on Tuesday June 2
- LENGTH: 1 hour and 50 minutes in length.
- You may bring in one (1) “cheat sheet”:
  - Must be HAND WRITTEN with pen/pencil on said sheet of paper (not typed, not photo copied, not printed, not written on an iPad)
  - Must be on 8.5 by 11 inch sheet of paper or smaller
  - You may write on both sides
  - No magnifying glasses or anything else silly
  - **I will confiscate cheatsheets that do not follow these rules** 🥀
  - I do not care what is written on it
- Exam is hand written on paper, bring something to write with
- You may bring a non-programmable, non-graphing calculator.

To do:

- Read Chapters 2.3 before Friday's class
- Assignment 1 due TONIGHT 11:59pm. Submit on Gradescope.