

Lecture 6

Random Variables and Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a random variable
- Define and calculate distributions
- Use distributions to calculate probabilities

1 Random Variables

Review

- So far, we have discussed how probability measures (functions) operate on events (subsets) of the sample space.
- For any event $A \subseteq \Omega$, $\mathbb{P}(A)$ is a number between 0 and 1.
- But events are not always the most natural way to talk about possible outcomes of an experiment.
- It is often easier to describe an event $A \subseteq \Omega$ with a single number.

EXAMPLE

- Recall our experiment “roll a die until you see 6”.
- There are many sequences of rolls (elements of Ω) in the event “first 6 on roll 12”: E_{12} .
- What if we summarize these with “12”?

Random Variables

- A random variable is a way of summarizing events mathematically.
- We use capital letters (usually, near the end of alphabet) to denote random variables (X, Y, Z , etc.)

DEFINITION

Random variable: a function from the sample space to a subset of the real numbers.

Therefore, a the random variable called X is any function

$$X : \underline{\Omega} \rightarrow \underline{\mathbb{R}}.$$

That is, for each $\omega \in \Omega$,

$$X(\omega) \in \mathbb{R}.$$

Random Variables

💡 EXAMPLE

Suppose we toss a coin 3 times.

- Sample space: $\Omega = \left\{ (x_1, x_2, x_3) : x_i \in \{H, T\} \right\}$

Let $X(\omega)$ = number of heads in ω .

- If $\omega = (HHH)$, $X(\omega) = .$ **3**
- If $\omega = (HHT), (HTH)$, or (THH) , $X(\omega) = .$ **2**
- ... and so on

We can summarize this as:

$$X(\omega) = \begin{cases} 0 & \omega = TTT \\ 1 & \omega = HTT, TTH, THT \\ 2 & \omega = HHT, HTH, THH \\ 3 & \omega = HHH \end{cases}$$

Random Variables

EXERCISE: WEATHER

Suppose it can either be raining, snowy, or sunny.

Let $X = 3$ if its raining, $X = 6$ if it snows, and $X = -2.7$ if it's sunny.

Is X a random variable?

YES

$$X(\omega) = \begin{cases} 3 & \omega = \text{rain} \\ 6 & \omega = \text{snow} \\ -2.7 & \omega = \text{sunny} \end{cases}$$

is defined for all $\omega \in \Omega$
and maps $\Omega \rightarrow \mathbb{R}$

Random Variables

💡 EXAMPLE

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $X(\omega) = \omega$ and $Y(\omega) = \omega^3 + 2$.

Compute $Z(\omega)$ for all $\omega \in \Omega$ where $Z = XY$.

$$Z(\omega) = X(\omega)Y(\omega) = (\omega)(\omega^3 + 2) = \begin{cases} (1)(1^3 + 2) = 3 & \omega = 1 \\ (2)(2^3 + 2) = 20 & \omega = 2 \\ (3)(3^3 + 2) = 87 & \omega = 3 \\ \vdots & \\ (6)(6^3 + 2) = 1308 & \omega = 6 \end{cases}$$

Canucks Hockey

- Suppose the Canucks play 2 games next week.
- Each game can be won (W), lost (L), or lost in overtime (O).
- The Canucks get points for wins (2), overtime losses (1), and losses (0).

EXERCISE:

- Write out the sample space Ω .
- Define $Z = \#$ games played. Is Z a random variable? If so, enumerate how it operates on each $\omega \in \Omega$
- Define $X = \#$ points. Enumerate how it operates on each $\omega \in \Omega$.
- Which outcomes are in the event $A = \{X \geq 3\}$?

a) $\Omega = \{WW, LL, OO, WL, LW, WO, OW, LO, OL\}$

b) $Z(\omega) = 2 \quad \forall \omega \in \Omega - \quad Z: \Omega \rightarrow \mathbb{R}$ so yes, this is a RV.

Canucks Hockey

$$c) X(w) = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{cases}$$

$w = LL$

$w = LO, OL$

$w = WL, LW, OO$

$w = WO, OW$

$w = WW$

$W = 2 \text{ pt}, O = 1 \text{ pt}, L = 0 \text{ pt}$

$$d) A = \{X \geq 3\} = \{WO, OW, WW\}$$

Random Variables and Events

- Random variables are naturally used to describe events of interest.
- For example $\{X > 3\}$ or $\{Y \leq 2\}$
- Formally, this notation means:

$$\{X > 3\} = \{\omega \in \Omega : X(\omega) > 3\},$$

and

$$\{Y \leq 2\} = \{\omega \in \Omega : Y(\omega) \leq 2\}.$$

In other words, $\{X > 3\}$ and $\{Y \leq 2\}$ are events.

The Indicator Function

The indicator function is a useful mathematical shorthand.

It is also a random variable: it is a function from $\Omega \rightarrow \mathbb{R}$.

DEFINITION

The indicator of the event A is the random variable given by

$$\underline{I}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{else.} \end{cases}$$

↳ is $\omega \in A$? if yes, = 1
no, = 0

$\underline{1}_A(\omega)$

The Indicator Function

💡 EXAMPLE

Let

$$A = \{1, 2, 3, 4, 5, 6\}$$

$I_A(5) = 1$ because 5 is in the set A.

$I_A(9) = 0$ because 9 is not in the set A.

The Indicator Function

💡 EXAMPLE

Consider a 6-sided fair die, where $\Omega = \{1, 2, 3, 4, 5, 6\}$, $X(\omega) = \omega$.

What is $Y = X + I_{\{6\}}$.?

$$\begin{aligned} Y(\omega) &= X(\omega) + I_{\{6\}}(\omega) \\ &= \omega + \underbrace{I_{\{6\}}(\omega)}_{\substack{=1 \text{ when } \omega=6 \\ =0 \text{ otherwise}}} \end{aligned}$$

$$Y(\omega) = \begin{cases} \omega & , \omega = 1, 2, 3, 4, 5 \\ 6+1=7 & \omega = 6 \end{cases}$$

$$Y(1) = 1, \quad Y(4) = 4, \quad Y(6) = 7$$

The Indicator Function

EXERCISE: PRODUCT OF INDICATORS

Let A and B be events, and let $X = I_A \times I_B$. Is X an indicator function? If so, of what event?

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

$$I_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & \omega \notin B \end{cases}$$

$$X(\omega) = \begin{cases} 1 & \text{when } I_A(\omega) = 1 \text{ and } I_B(\omega) = 1 \\ & \Rightarrow \omega \in A \text{ and } \omega \in B \\ & \Rightarrow \omega \in A \cap B \\ 0 & \omega \notin A \cap B \end{cases}$$

X indicates whether or not $\omega \in A \cap B$

$$X = I_{(A \cap B)}$$

2 Distributions

Distributions

- A probability associates each $\omega \in \Omega$ with a number between 0 and 1 and satisfies the Probability axioms.
- Random variables also operate on Ω . Taking these together **induces** a probability on $\mathbb{R}$.

EXAMPLE

Suppose we flip a fair coin with $\Omega = \{H, T\}$, and we know that $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$.

Then, considering the RV I_H , we see $\mathbb{P}(I_H = 1) = \mathbb{P}(\{\omega : I_H(\omega) = 1\}) = \mathbb{P}(H)$

Distributions

DEFINITION

Distribution: If X is a random variable, then the **distribution** of X is the collection of probabilities $\mathbb{P}(X \in B)$ for all subsets B of $\mathbb{R}$.

Consider $\Omega = \{H, T\}$, and $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$

- This defines $\mathbb{P}$ on all subsets $2^\Omega = \{\emptyset, \{T\}, \{H\}, \Omega\}$.
- But considering a random variable X , it isn't really enough to know it's probability only on it's range.
- We need to know more than that: we need to know, for **every** $B \subset \mathbb{R}$, what is $\mathbb{P}(X \in B) = \mathbb{P}(\{s \in \Omega : X(s) \in B\})$.
- To fully specify the distribution, we need to do this for every (nice) subset B . The collection of these subsets is denoted $\mathcal{B}$.

The mathematical details are at least at the level of Math 420.

Distribution Example

💡 EXAMPLE

Consider shuffling a deck of 50 Pokemon cards where 20 are grass-type (G), 13 are fighting-type (F), 7 are water-type (W), and 10 are electric-type (E). Professor Grace really likes the grass and water types.

Consider the random variable Z , which indicates if Grace will *really like* the card chosen. Compute $\mathbb{P}(Z = z)$ for any $z \in \mathbb{R}$

$$P(w) = \begin{cases} \frac{20}{50} = 0.4 & w = G \\ 0.26 & w = F \\ 0.14 & w = W \\ 0.20 & w = E \end{cases}$$

$$\begin{aligned} Z &= I_{\{\text{Grass or water types}\}} \\ &= I_{GUW} \end{aligned}$$

⇒ Possible values for Z are 1 or 0

Distribution Example

$$\begin{aligned}P(Z=0) &= P(\{\omega : Z(\omega)=0\}) \\&= P(\text{not grass or water}) \\&= P(E \cup F) \\&= P(E) + P(F) \\&= 0.20 + 0.26 \\&= 0.46\end{aligned}$$

$$\begin{aligned}P(Z=1) &= P(\{\omega : Z(\omega)=1\}) \\&= P(G \cup W) \\&= P(G) + P(W) \\&= 0.40 + 0.14 \\&= 0.54\end{aligned}$$

$$P(Z=z) = \begin{cases} 0.46 & z=0 \\ 0.54 & z=1 \\ 0 & \text{otherwise} \end{cases}$$



must include P for all
 $z \in \mathbb{R}$

distribution: $\mathbb{R} \rightarrow [0,1]$

(random var) $\rightarrow$ probability

More Hockey

- Suppose the Canucks play 2 games next week. Each game can be won (W), lost (L), or lost in overtime (O). The Canucks get points for wins (2), overtime losses (1), and losses (0).
- The Canucks are **not very good**. Suppose for each game (independently),

$$\mathbb{P}(\omega) = \begin{cases} 0.3 & \omega = W \\ 0.2 & \omega = O \\ 0.5 & \omega = L. \end{cases}$$

EXERCISE: SHOW THE CANUCKS ARE NOT VERY GOOD

- a. Let X be the number of wins. Compute $\mathbb{P}(X = x)$ for all $x \in \mathbb{R}$. *Hint: Start by solving for $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, and $\mathbb{P}(X = 2)$*
- b. Write a formula for $\mathbb{P}(X \in B)$ for any $B \subset \mathbb{R}$. *Hint: indicators!*
- c. Let Y be the number of points. Compute $\mathbb{P}(Y = y)$ for all $y \in \mathbb{R}$.

More Hockey

$X = \#$ of wins (of two games)

$$P(L)P(L) + P(O)P(O) + 2P(L)P(O)$$

$$P(X=0) = P(\{\omega : X(\omega)=0\}) = P(\{LL, OO, LO, OL\})$$

$$= P(\text{not win first game AND not win second})$$

$$= P(W^c)P(W^c)$$

$$= (1-0.3)(1-0.3) = 0.49$$

$$P(X=1) = P(\{\omega : X(\omega)=1\}) = P(\{WL, LW, WO, OW\})$$

$$= P(W)P(L) + P(L)P(W) + P(W)P(O) + P(O)P(W)$$

$$= 2(0.3)(0.5) + 2(0.2)(0.3)$$

$$= 0.42$$

$$P(X=2) = P(\{WW\}) = (0.3)(0.3) = 0.09$$

More Hockey

$$a) \quad P(X=x) = \begin{cases} 0.49 & x=0 \\ 0.42 & x=1 \\ 0.09 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

$$b) \quad P(X \in B) = 0.49 I_B(0) + 0.42 I_B(1) + 0.09 I_B(2)$$

c)

More Hockey

c)

$$P(Y=0) = P(\{\omega : Y(\omega) = 0\}) = P(LL) = 0.5 \cdot 0.5 = 0.25$$

$$P(Y=1) = P(\{\omega : Y(\omega) = 1\}) = P(\{LO, OL\}) = 2(0.5)(0.2) = 0.2$$

⋮

$$P(Y=4) =$$

then,

$$P(Y=y) = \begin{cases} 0.25 & y=0 \\ 0.20 & y=1 \\ 0.34 & y=2 \\ 0.12 & y=3 \\ 0.09 & y=4 \\ 0 & \text{otherwise} \end{cases}$$

Upcoming Assessments:

- Checkpoint 2 (Section 1.5: Conditional Probability and Independence) is on Friday Oct 02
- Checkpoint 3 (Sections 2.1 - 2.3: Random Variables, Distributions, and Discrete Distributions) is on Friday Oct 09
- Midterm 1 (Sections 1.1 - 2.4) is on Friday Oct 16