

Lecture 6

Random Variables and Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a random variable
- Define and calculate distributions
- Use distributions to calculate probabilities

1 Random Variables

Review

- So far, we have discussed how probability measures (functions) operate on events (subsets) of the sample space.
- For any event $A \subseteq \Omega$, $\mathbb{P}(A)$ is a number between 0 and 1.
- But events are not always the most natural way to talk about possible outcomes of an experiment.
- It is often easier to describe an event $A \subseteq \Omega$ with a single number.

EXAMPLE

- Recall our experiment “roll a die until you see 6”.
- There are many sequences of rolls (elements of Ω) in the event “first 6 on roll 12”: E_{12} .
- What if we summarize these with “12”?

Random Variables

- A random variable is a way of summarizing events mathematically.
- We use capital letters (usually, near the end of alphabet) to denote random variables (X, Y, Z , etc.)

DEFINITION

Random variable: a **function** from the sample space to a subset of the real numbers.

Therefore, a the random variable called X is any function

$$X : \Omega \rightarrow \mathbb{R}.$$

That is, for each $\omega \in \Omega$,

$$X(\omega) \in \mathbb{R}.$$

Random Variables

💡 EXAMPLE

Suppose we toss a coin 3 times.

- Sample space: $\Omega = \left\{ (x_1, x_2, x_3) : x_i \in \{H, T\} \right\}$

Let $X(\omega)$ = number of heads in ω .

- If $\omega = (HHH)$, $X(\omega) = 3$.
- If $\omega = (HHT)$, (HTH) , or (THH) , $X(\omega) = 2$.
- ... and so on

We can summarize this as:

Random Variables

EXERCISE: WEATHER

Suppose it can either be raining, snowy, or sunny.

Let $X = 3$ if its raining, $X = 6$ if it snows, and $X = -2.7$ if it's sunny.

Is X a random variable?

Random Variables

EXAMPLE

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $X(\omega) = \omega$ and $Y(\omega) = \omega^3 + 2$.

Compute $Z(\omega)$ for all $\omega \in \Omega$ where $Z = XY$.

Canucks Hockey

- Suppose the Canucks play 2 games next week.
- Each game can be won (W), lost (L), or lost in overtime (O).
- The Canucks get points for wins (2), overtime losses (1), and losses (0).

EXERCISE:

- a. Write out the sample space Ω .
- b. Define $Z = \#$ games played. Is Z a random variable? If so, enumerate how it operates on each $\omega \in \Omega$
- c. Define $X = \#$ points. Enumerate how it operates on each $\omega \in \Omega$.
- d. Which outcomes are in the event $A = \{X \geq 3\}$?

Canucks Hockey

Random Variables and Events

- Random variables are naturally used to describe events of interest.
- For example $\{X > 3\}$ or $\{Y \leq 2\}$
- Formally, this notation means:

$$\{X > 3\} = \{\omega \in \Omega : X(\omega) > 3\},$$

and

$$\{Y \leq 2\} = \{\omega \in \Omega : Y(\omega) \leq 2\}.$$

In other words, $\{X > 3\}$ and $\{Y \leq 2\}$ are events.

The Indicator Function

The indicator function is a useful mathematical shorthand.

It is also a random variable: it is a function from $\Omega \rightarrow \mathbb{R}$.

DEFINITION

The indicator of the event A is the random variable given by

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \text{else.} \end{cases}$$

The Indicator Function

EXAMPLE

Let

$$A = (1, 2, 3, 4, 5, 6)$$

$I_A(5) = 1$ because 5 is in the set A.

$I_A(9) = 0$ because 9 is not in the set A.

The Indicator Function

EXAMPLE

Consider a 6-sided fair die, where $\Omega = \{1, 2, 3, 4, 5, 6\}$

What is $Y = X + I_{\{6\}}$?

The Indicator Function

EXERCISE: PRODUCT OF INDICATORS

Let A and B be events, and let $X = I_A \times I_B$. Is X an indicator function? If so, of what event?

2 Distributions

Distributions

- A probability associates each $\omega \in \Omega$ with a number between 0 and 1 and satisfies the Probability axioms.
- Random variables also operate on Ω . Taking these together **induces** a probability on $\mathbb{R}$.

EXAMPLE

Suppose we flip a fair coin with $\Omega = \{H, T\}$, and we know that $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$.

Then, considering the RV I_H , we see $\mathbb{P}(I_H = 1) = \mathbb{P}(\{\omega : I_H(\omega) = 1\}) = \mathbb{P}(H)$

Distributions

DEFINITION

Distribution: If X is a random variable, then the **distribution** of X is the collection of probabilities $\mathbb{P}(X \in B)$ for all subsets B of $\mathbb{R}$.

Consider $\Omega = \{H, T\}$, and $\mathbb{P}(H) = \mathbb{P}(T) = 0.5$

- This defines $\mathbb{P}$ on all subsets $2^\Omega = \{\emptyset, \{T\}, \{H\}, \Omega\}$.
- But considering a random variable X , it isn't really enough to know it's probability only on it's range.
- We need to know more than that: we need to know, for **every** $B \subset \mathbb{R}$, what is $\mathbb{P}(X \in B) = \mathbb{P}(\{s \in \Omega : X(s) \in B\})$.
- To fully specify the distribution, we need to do this for every (nice) subset B . The collection of these subsets is denoted $\mathcal{B}$.

The mathematical details are at least at the level of Math 420.

Distribution Example

EXAMPLE

Consider shuffling a deck of 50 Pokemon cards where 20 are grass-type (G), 13 are fighting-type (F), 7 are water-type (W), and 10 are electric-type (E). Professor Grace really likes the grass and water types.

Consider the random variable Z , which indicates if Grace will *really like* the card chosen. Compute $\mathbb{P}(Z = z)$ for any z .

Distribution Example

More Hockey

- Suppose the Canucks play 2 games next week. Each game can be won (W), lost (L), or lost in overtime (O). The Canucks get points for wins (2), overtime losses (1), and losses (0).
- The Canucks are **not very good**. Suppose for each game (independently),

$$\mathbb{P}(\omega) = \begin{cases} 0.3 & \omega = W \\ 0.2 & \omega = O \\ 0.5 & \omega = L. \end{cases}$$

EXERCISE: SHOW THE CANUCKS ARE NOT VERY GOOD

- Let X be the number of wins. Compute $\mathbb{P}(X = x)$ for all $x \in \mathbb{R}$. *Hint: Start by solving for $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, and $\mathbb{P}(X = 2)$*
- Write a formula for $\mathbb{P}(X \in B)$ for any $B \subset \mathbb{R}$. *Hint: indicators!*
- Let Y be the number of points. Compute $\mathbb{P}(Y = y)$ for all $y \in \mathbb{R}$.

More Hockey

More Hockey

More Hockey

Upcoming Assessments:

- Checkpoint 2 (Section 1.5: Conditional Probability and Independence) is on Friday Oct 02
- Checkpoint 3 (Sections 2.1 - 2.3: Random Variables, Distributions, and Discrete Distributions) is on Friday Oct 09
- Midterm 1 (Sections 1.1 - 2.4) is on Friday Oct 16