

Lecture 7

Discrete Random Variables (Part 1)

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Identify discrete distributions
- Use the Bernoulli and binomial distributions to solve for probabilities

1 Discrete Random Variables

Discrete Random Variables

DEFINITION

We say that a random variable X is **discrete** if there exists a countable set $K = \{x_1, x_2, \dots\}$ such that

$$\mathbb{P}(X \in K) = 1.$$

- In particular, this means that $\{x : P(X = x) > 0\}$ is countable:

I can count the x that have positive probability.

- For discrete RVs, we call the set $\{x : P(X = x) > 0\}$ the **support** of X .

Examples of Experiments Described by Discrete Random Variables

- Toss a fair coin 3 times, count the total number of Heads.
- Roll a 6-sided die until you see 6, count the number of rolls.
- Count the number of people that arrive at the bus stop in some amount of time.
- Will it be rainy (-6), cloudy (2), or sunny ($+10$) on campus today?

Probability Mass Function (PMF)

- For discrete random variables, we can be explicit about the probability associated with each value in its support

DEFINITION

For a discrete random variable, its probability (mass) function is the function $p_X : \mathbb{R} \rightarrow [0, 1]$ defined by

$$p_X(x) = \mathbb{P}(X = x).$$

- ↑ ↑

It is occasionally written $f_X(x) = \mathbb{P}(X = x)$.

- Each of the examples of discrete random variables has a PMF. Once we write it down, we know everything there is to know.

A Rainy Day

EXERCISE: RAINY DAYS

Suppose it rains 30% of the time, is cloudy but not raining 30% of the time, and sunny 40% of the time. Your happiness is given by the random variable Z with $Z = -6, 2$, and 10 respectively.

- a. What is the support of Z ?
- b. What is the PMF of Z ?

a) The values of z with non-zero probability are $-6, 2, 10$.

$$P(Z=z) = \begin{cases} 0.30 & \text{when } z = -6 \\ 0.30 & z = 2 \\ 0.40 & z = 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P(Z=z) = 0.3 I_{\{-6\}}(z) + 0.3 I_{\{2\}}(z) + 0.4 I_{\{10\}}(z)$$

A Rainy Day

Fair Coins

EXERCISE: TOSS A COIN 3 TIMES

Toss a fair coin 3 times. Let X be the number of heads.

- a. What is the support of X ? *→ what values of X are possible?*
- b. What is the $p_X(x)$?

a) 0, 1, 2,

b) $p_X(x) = P(X=x) =$

$\frac{1}{8}$	$x=0$
$\frac{3}{8}$	$x=1$
$\frac{3}{8}$	$x=2$
$\frac{1}{8}$	$x=3$
<u>0</u>	otherwise

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Finding Probabilities of Events

The PMF can be used to find the probabilities of events.

We have that for some event A and a RV X ,

$$\mathbb{P}(X \in A) = \sum_{x \in A} p_X(x).$$

💡 EXAMPLE

Recall: it rains 30% of the time, is cloudy but not raining 30% of the time, and sunny 40% of the time. Your happiness is given by the random variable Z with $Z = -6, 2$, and 10 respectively.

- Let A be the event that you are happier than 0. Find $\mathbb{P}(Z \in A)$.

$$p_Z(z) = \begin{cases} 0.3 & z = -6 \\ 0.3 & z = 2 \\ 0.4 & z = 10 \\ 0 & \text{o/w} \end{cases}$$

$$\begin{aligned} \mathbb{P}(Z \in A) &= \sum_{z \in A} p_Z(z) = \mathbb{P}(Z > 0) = \mathbb{P}(Z = 2) + \mathbb{P}(Z = 10) \\ &= 0.3 + 0.4 \\ &= 0.7 \end{aligned}$$

Another PMF

For a fixed number $\theta \in [0, 1]$, define a discrete random variable X with PMF given by:

$$p_X(x; \theta) = \begin{cases} (1 - \theta)^2 & x = 0, \\ 2\theta(1 - \theta) & x = 1, \\ \theta^2 & x = 2, \\ 0 & \text{else.} \end{cases}$$

- Different values of θ will give different PMFs.

$$p_X(x; \underline{\theta = 0.1}) = \begin{cases} 0.81 & x = 0, \\ 0.18 & x = 1, \\ 0.01 & x = 2 \\ 0 & \text{else.} \end{cases}$$

By including a parameter θ , we are able to express many distributions (a family of distributions) with a single functional form.

2 Discrete families

Bernoulli

An experiment with 2 outcomes (success/failure).

DEFINITION

Suppose the probability of success is $\theta \in (0, 1)$. A RV X with PMF given by

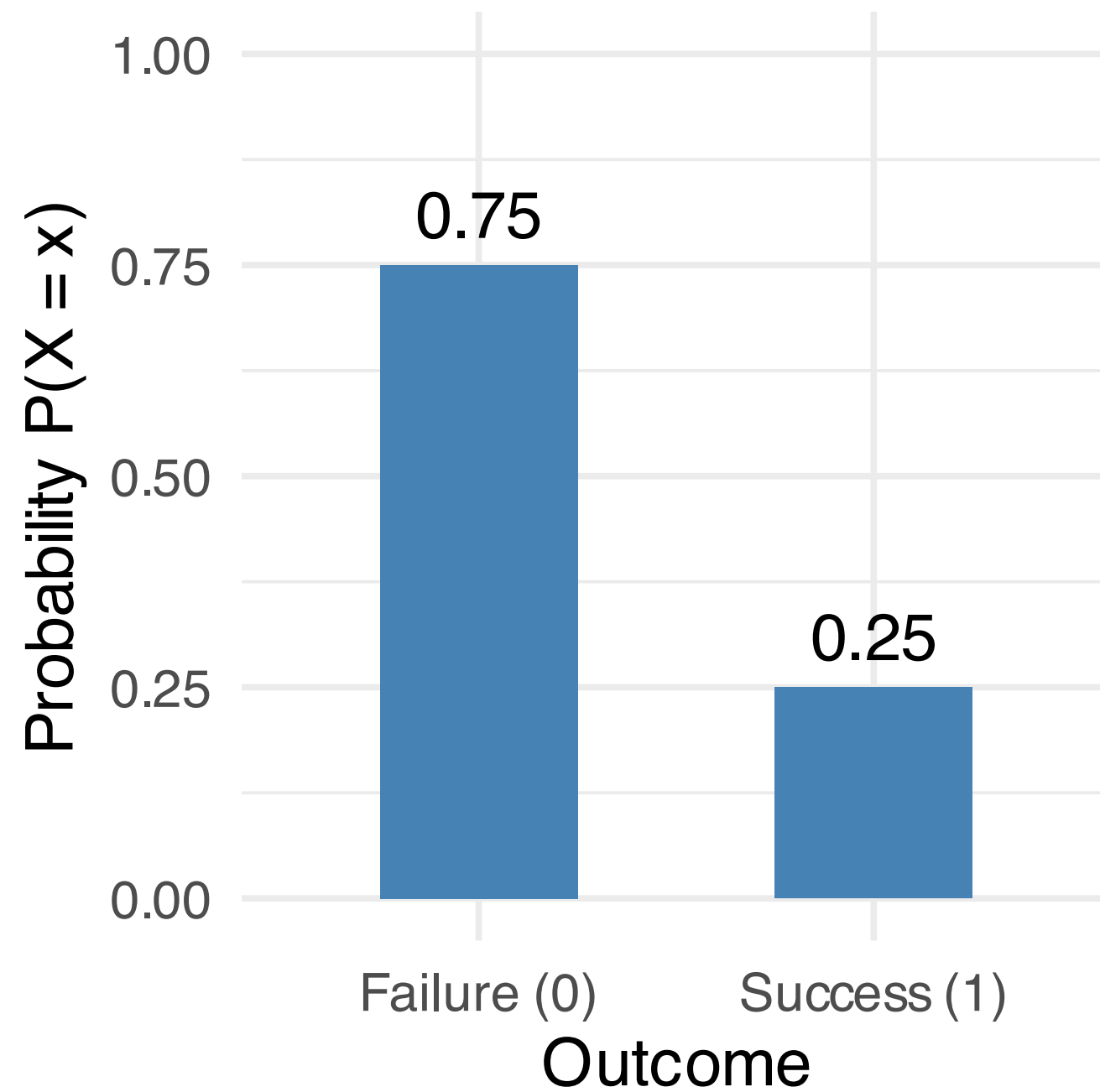
$$\underline{p_X(x; \theta)} = \theta^x (1 - \theta)^{1-x} I_{\{0,1\}}(x),$$

is said to have the $\text{Bern}(\theta)$ distribution.

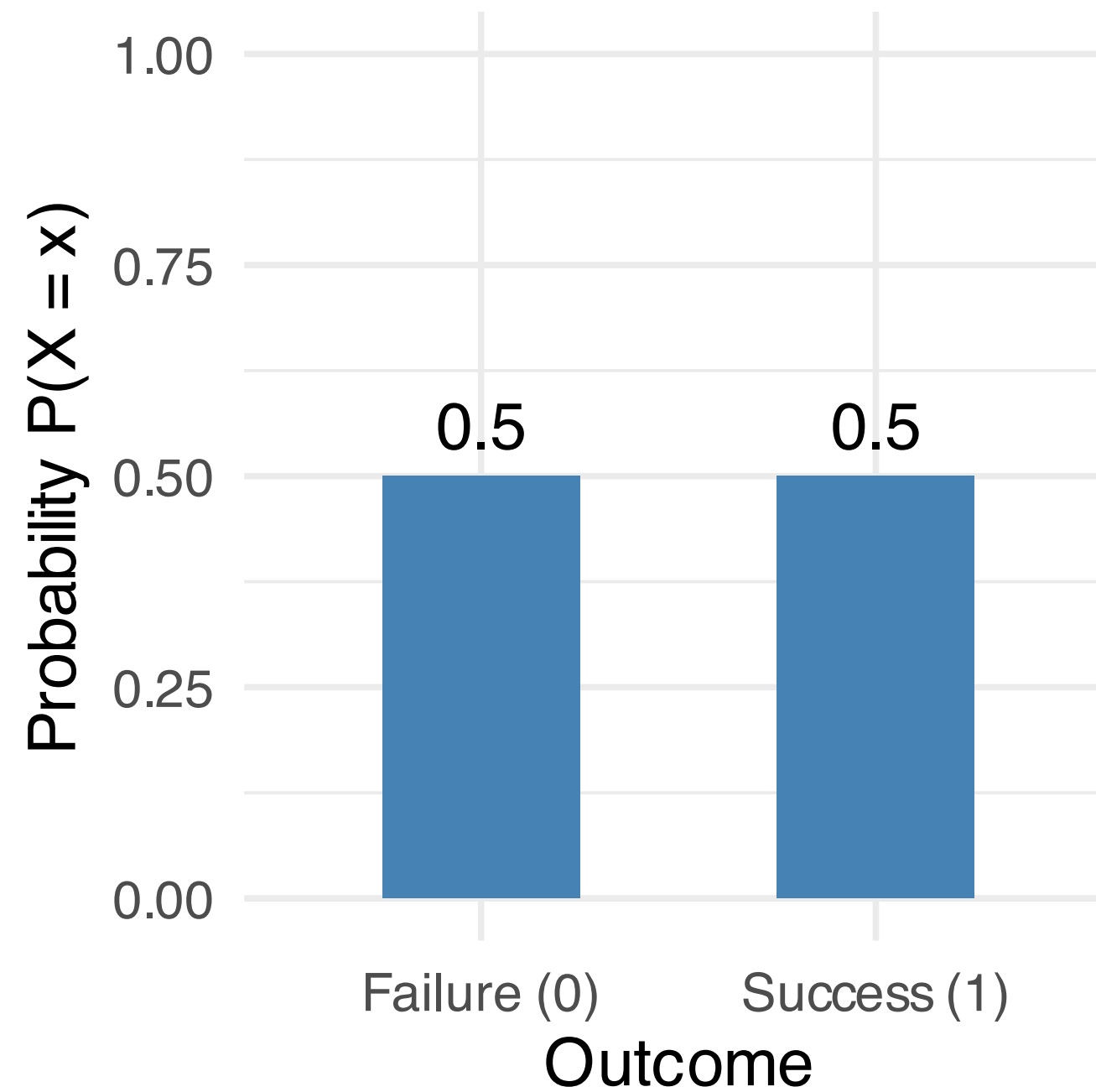
- If $\theta = 0.5$, then we have our favourite fair-coin friend.
- But we can let θ be numbers other than 0.5.

Bernoulli

Bernoulli PMF ($\theta = 0.25$)



Bernoulli PMF ($\theta = 0.5$)



Binomial

Used to determine the number of “successes” (x) in a sequence of n independent Bernoulli trials, each with the same probability of success (θ).

📖 DEFINITION

Suppose the probability of success is $\theta \in (0, 1)$. A RV X with PMF given by

$$p_X(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} I_{\{0,1,\dots,n\}}(x),$$

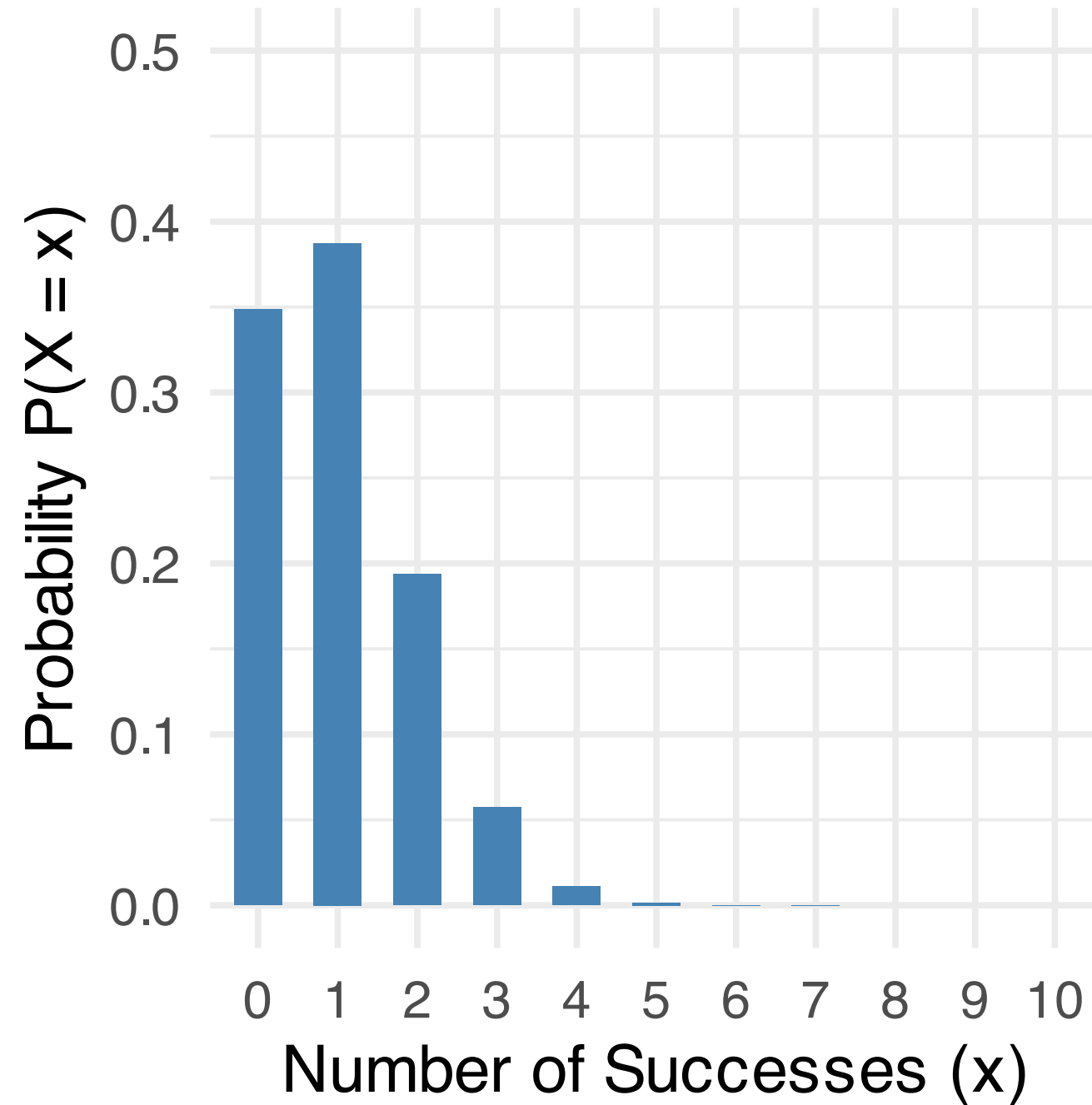
is said to have the $\text{Binom}(n, \theta)$ distribution.

forcing the successes with non-zero probability to be between 0 and n .

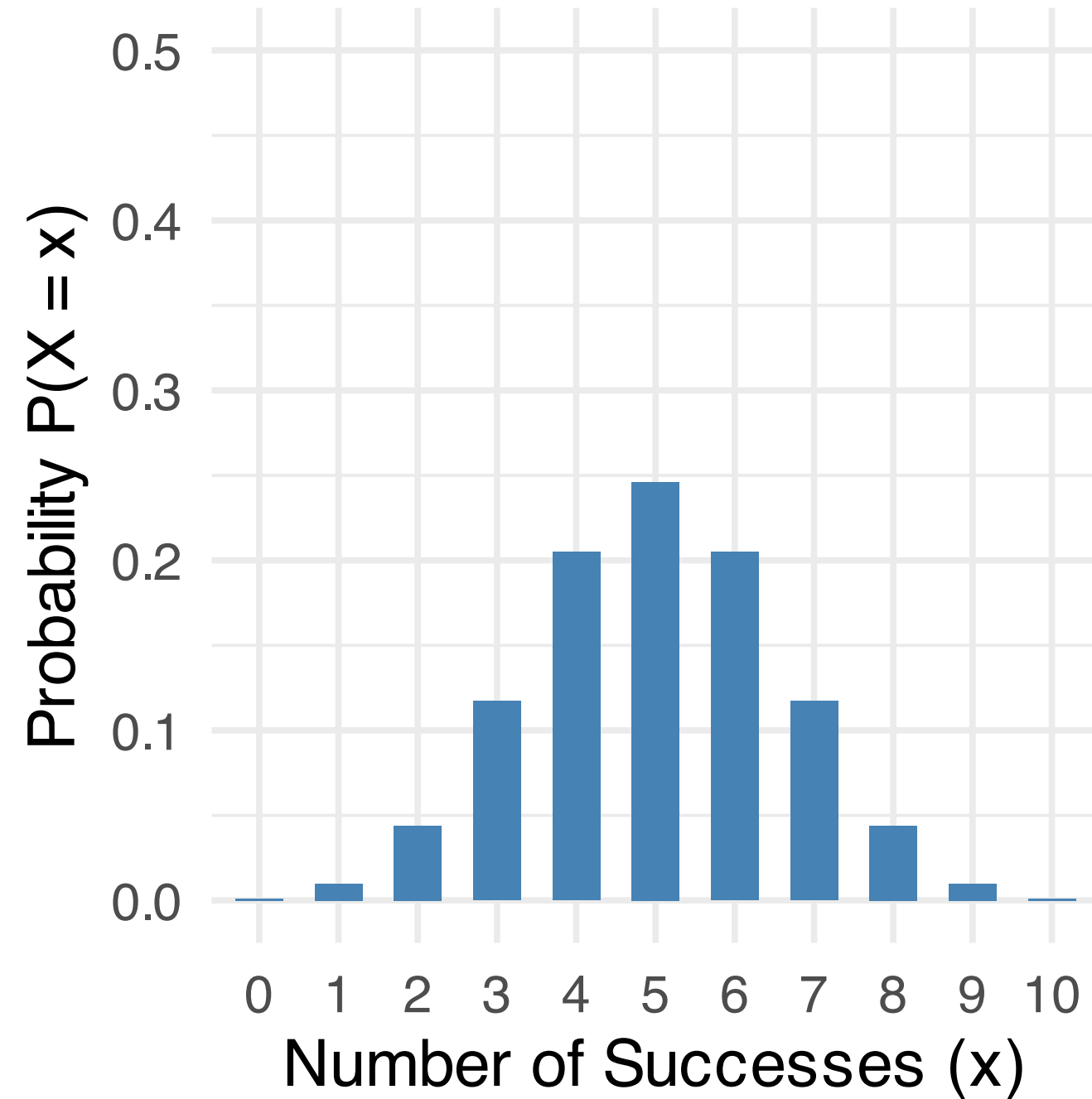
$$p_X(x) = \begin{cases} \binom{n}{x} \theta^x (1 - \theta)^{n-x} & \text{if } x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Binomial

Binomial PMF ($\theta = 0.1, n = 10$)



Binomial PMF ($\theta = 0.5, n = 10$)



Using the Binomial

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$$p_X(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x} I_{\{0,1,\dots,n\}}(x)$$

EXERCISE: GUESSING ON THE MIDTERM

a. Suppose Midterm 1 has 5 True/False questions. What is the probability that you get at least 4 correct answers by random guessing? $\theta = 0.5$ $n = 5$

b. Suppose Midterm 1 has 2 multiple choice questions with 5 options. What is the probability that you get both correct by random guessing? $\theta = \frac{1}{5} = 0.2$ | $n = 2$ $P(X=2)$

Using the Binomial

a) $P(\text{success}) = \theta = 0.5$

$$n = 5$$

$X = \# \text{ get right (success)}$

$$P(X \geq 4) = P(X=4) + P(X=5)$$

$$= \binom{5}{4} 0.5^4 (1-0.5)^{5-4} + \binom{5}{5} 0.5^5 (1-0.5)^{5-5}$$

$$= 5 \cdot 0.5^4 \cdot 0.5^1 + (1) 0.5^5 \cdot 0.5^0$$

$$= 0.1875$$

$$\theta = 0.2$$

$$n = 2$$

b) $X = 2$

$$P(X=2) = \binom{2}{2} 0.2^2 (1-\theta)^{2-2}$$

$$= (1) (0.2)^2$$

$$= 0.04$$

Checkpoint 2

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing calculator
- Writing materials

Phones and laptops put away, please! You can have 15 minutes.