Lecture 7

Continuous Random Variables

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a continuous random variable
- Define and identify a probability density function
- Define, identify, and apply common families of continuous distributions

1 Continuous Random Variables

Continuous Random Variables

📖 DEFINITION

A RV X is **continuous** if

$$\mathbb{P}(X = x) = 0,$$

for every $x \in \mathbb{R}$.

- This is the most rigorous way to define continuous RVs, but it doesn't provide much intuition.
- We need a bit more to make our intuition match the math.

Absolutely Continuous RVs and Density Functions

📖 **DEFINITION**

A function $f : \mathbb{R} \to \mathbb{R}$ is called a **density function** if $f(x) \geq 0, \ \forall x \in \mathbb{R}$ and $\int f(x)\mathrm{d}x = 1$.

📖 **DEFINITION**

A RV X is **absolutely continuous** if there exists a **density function** f such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x$$

whenever $a \leq b$.

We call such a density function a **probability density function** (PDF) and will often use the notation $f_X(x)$ to denote its relationship to X.

Absolutely Continuous RVs

📐 **THEOREM**

Let X be an absolutely continuous random variable. Then X is a continuous random variable (i.e., $\mathbb{P}(X = a) = 0$ for all $a \in \mathbb{R}$)

📝 **PROOF**

$\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)dx = 0$

ⓘ **Note**

Note: the converse is not necessarily true. That is, not all continuous distributions you will come across in statistics (in general) are **absolutely** continuous. We will stick to absolutely continuous distributions in this course.

Comparison to Discrete RVs

- Consider some set $A \subset \mathbb{R}$.
- Let X be a random variable.

Probability of A for discrete RV:

$$\mathbb{P}(X \in A) = \sum_{x \in A} p_X(x).$$

Probability of A for an (absolutely) continuous RV:

$$\mathbb{P}(X \in A) = \int_A f_X(x)\mathrm{d}x.$$

Example

✏️ **EXERCISE: EXAMPLE CONTINUOUS RV**

Let X be a RV with PDF

$$f_X(x) = 2x^{-3} I_{[1,\infty)}(x).$$

a. Is X absolutely continuous? What is $f_X(2)$?

b. What is $\mathbb{P}(0 < X < 2)$?

c. Suppose we defined $g(x) = 2x^{-3}$ (no indicator function). Is $g(x)$ a density function? Does is have the same support?

Example

2 Continuous families

(Continuous) Uniform

📖 **DEFINITION**

Let $L < R \in \mathbb{R}$. A RV X with PDF

$$f_X(x; L, R) = \frac{1}{R - L} I_{[L,R]}(x),$$

is said to have the $\mathrm{Unif}(L, R)$ distribution.

Continuous Uniform Median

✏️ **EXERCISE: UNIFORM MEDIAN**

For continuous random variables, the **median** is the number m such that

$$\mathbb{P}(X < m) = \mathbb{P}(X > m) = 1/2.$$

Let $X \sim \text{Unif}(L, R)$. What is the median of X?

Continuous Uniform Median

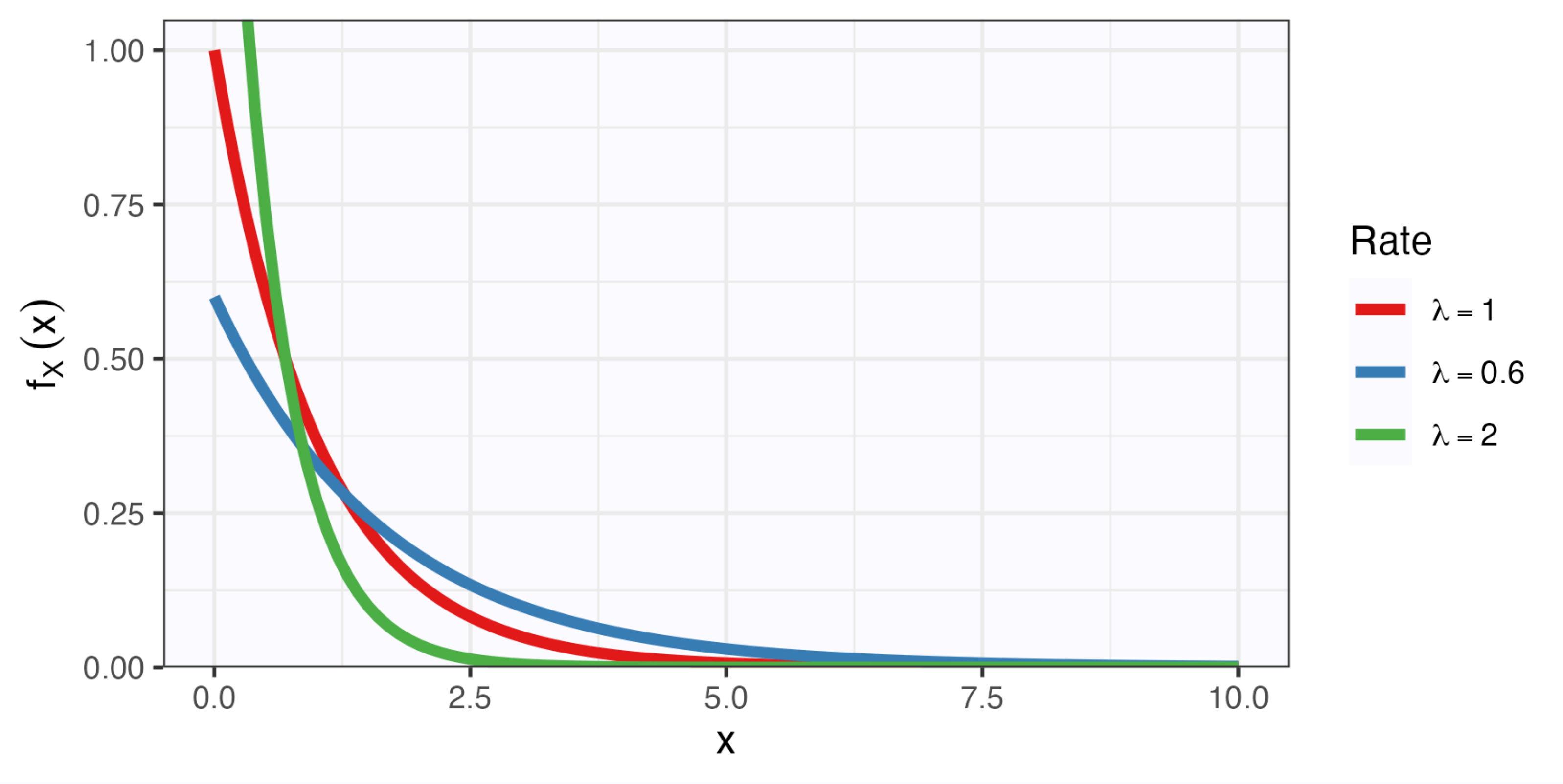
Exponential Distribution

📖 DEFINITION

Let $\lambda > 0$. A RV X with PDF

$$f_X(x;\lambda) = \lambda e^{-\lambda x} I_{[0,\infty)}(x),$$

is said to have the $\mathrm{Exp}(\lambda)$ distribution with rate λ.



Exponential Distribution

$$f_X(x;\lambda) = \lambda e^{-\lambda x} I_{[0,\infty)}(x)$$

✏️ **EXERCISE: EXPONENTIAL**

Let $Y \sim \mathrm{Exp}(2)$. Find $\mathbb{P}(Y > y)$ for $y > 0$.

Exponential Distribution

Exponential Distribution

✏️ **EXERCISE: EXPONENTIAL RATES**

A small coffee shop receives customers independently at an average rate of 12 per hour. Let T be the waiting time (in minutes) between consecutive customer arrivals.

a. What distribution does T follow, and what is its rate parameter λ?

b. What is the probability that the next customer arrives within 3 minutes?

Exponential Distribution

Gamma Distribution

Recall the gamma function:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t)dt, . \text{ where } \alpha > 0$$

Useful results:

- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
- If α is an integer, this integral simplifies to $\Gamma(n) = (n-1)!$
- $\Gamma(1/2) = \sqrt{(}\pi)$

Gamma Distribution

📖 **DEFINITION**

Let $\alpha, \lambda > 0$. A RV Z with pdf

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0,\infty)}(z),$$

is said to have the $\text{Gam}(\alpha, \lambda)$ distribution with shape α and rate λ.

Note: $Z \sim \text{Gam}(1, \lambda) \Rightarrow Z \sim \text{Exp}(\lambda)$.

Gamma Distribution

✏️ **EXERCISE: GAS PUMPS**

The time to process an insurance claim follows $\mathrm{Gam}(\alpha = 2, \lambda = 1/3)$. What is the probability that the claim takes more than 6 hours to process?

Kernel and Integration Constant

$$f_Z(z;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0,\infty)}(z).$$

- PDFs/PMFs must integrate/sum to 1.
- The functional form can be thought of as two pieces:
  1. The “kernel” is the portion that depends on the argument (x or z)
  2. The “normalizing constant” is the part that depends only on parameters; this makes the function integrate to 1.
- The support (given by the indicator function) is part of the kernel.

Kernel and Integration Constant: Example

$$f_Z(z;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0,\infty)}(z)$$

1. The kernel is $z^\alpha e^{-\lambda z} I_{[0,\infty)}(z)$.
2. The normalizing constant is $\lambda^\alpha / \Gamma(\alpha)$.

We know that

$$1 = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} \mathsf{d}z \Longrightarrow \frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty z^{\alpha-1} e^{-\lambda z} \mathsf{d}z.$$

Kernel Matching

$$f_Z(z;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0,\infty)}(z)$$

✏️ **EXERCISE: KERNEL MATCHING**

1. What is

$$\int_0^\infty z^3 e^{-5z} \mathsf{d}z?$$

2. What is

$$\int_0^\infty z \frac{\lambda^4}{\Gamma(4)} z^3 e^{-\lambda z} \mathsf{d}z?$$

Hint: Recall that $\Gamma(n) = (n-1)!$ when $n \in \{1, 2, \ldots\}$.

Kernel Matching

The Normal (Gaussian) distribution

📖 DEFINITION

Let $\mu \in \mathbb{R}, \sigma > 0$. A RV Z with pdf

$$f_Z(z; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\},$$

is said to have the $\mathcal{N}(\mu, \sigma^2)$ distribution.

The Normal Distribution

- This distribution is incredibly important.
- The reason is that it is good for modelling averages. We'll justify this rigorously later.
- $Z \sim \mathcal{N}(0,1)$ is called the standard normal distribution. When Z is written without context, it is often understood to have this specific distribution.
- Unfortunately

$$\mathbb{P}(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z,$$

  does not have a closed form solution.
- Old folks (like me) used tables in textbooks to calculate this (Table D.2 on p. 712 for you).
- Nowadays, we use software.

To do:

- Read Chapter 2.5 before next class
- Assignment 2 due tomorrow, May 27th @ 11:59pm.
- Midterm is next Tuesday during class.