

Lecture 7

Continuous Random Variables

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a continuous random variable
- Define and identify a probability density function
- Define, identify, and apply common families of continuous distributions

1 Continuous Random Variables

Continuous Random Variables

DEFINITION

A RV X is continuous if

$$\mathbb{P}(X = x) = 0,$$

for every $x \in \mathbb{R}$.

- This is the most rigorous way to define continuous RVs, but it doesn't provide much intuition.
- We need a bit more to make our intuition match the math.

Absolutely Continuous RVs and Density Functions

DEFINITION

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called a density function if $f(x) \geq 0$, $\forall x \in \mathbb{R}$ and $\int f(x)dx = 1$.

DEFINITION

A RV X is absolutely continuous if there exists a density function f such that

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$$

whenever $a \leq b$.

We call such a density function a **probability density function** (PDF) and will often use the notation $f_X(x)$ to denote its relationship to X .

Absolutely Continuous RVs

THEOREM

Let X be an absolutely continuous random variable. Then X is a continuous random variable (i.e., $\mathbb{P}(X = a) = 0$ for all $a \in \mathbb{R}$)

PROOF

$$\mathbb{P}(X = a) = \mathbb{P}(\underline{a} \leq X \leq \underline{a}) = \int_a^a f(x) dx = 0$$

Note

Note: the converse is not necessarily true. That is, not all continuous distributions you will come across in statistics (in general) are **absolutely** continuous. We will stick to absolutely continuous distributions in this course.

Comparison to Discrete RVs

- Consider some set $A \subset \mathbb{R}$.
- Let X be a random variable.

Probability of A for discrete RV:

$$\mathbb{P}(X \in A) = \sum_{\underline{x \in A}} p_X(x).$$

Probability of A for an (absolutely) continuous RV:

$$\mathbb{P}(X \in A) = \int_{\underline{A}} f_X(x) dx.$$

Example

EXERCISE: EXAMPLE CONTINUOUS RV

Let X be a RV with PDF

$$f_X(x) = 2x^{-3}I_{[1,\infty)}(x).$$

Show $f_X(x)$ is a pdf.

a. Is ~~X absolutely continuous?~~ What is $f_X(2)$? $\leftarrow x=2$

b. What is $\mathbb{P}(0 < X < 2)$? What is $\mathbb{P}(0 \leq X \leq 2)$?

c. Suppose we defined $g(x) = 2x^{-3}$ (no indicator function). Is $g(x)$ a density function? Does it have the same support?

\rightarrow hint: the support of $f_X(x)$ is $[1, \infty)$

$$= \begin{cases} 2x^{-3}, & 1 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Example

$$f_X(x) = 2x^{-3} I_{[1, \infty)}(x)$$

a) Show valid pdf: ① $f(x) \geq 0 \forall x \in \mathbb{R}$, ② $\int f(x) dx = 1$

① if $x \in [1, \infty)$, $2x^{-3} \geq 0$
 $x \notin [1, \infty)$, $f(x) = 0$ $\therefore f(x) \geq 0 \forall x \in \mathbb{R}$ ✓

$$\textcircled{2} \int 2x^{-3} I_{[1, \infty)}(x) dx = \int_1^{\infty} 2x^{-3} dx = \frac{2x^{-2}}{-2} \Big|_{x=1}^{x=\infty} = 0 - (-1) = 1 \checkmark$$

$$f_X(2) = 2(2)^{-3} I_{[1, \infty)}(2) = (2)(2)^{-3}(1) = 1/4$$

$$b) P(0 < X < 2) = \int_0^2 f_X(x) dx = \int_0^2 2x^{-3} I_{[1, \infty)} dx = \int_1^2 2x^{-3} dx = 3/4$$

(this is also equal to $P(0 \leq X \leq 2)$)

Example

c) $g(x) = 2x^{-3}$.

No, without the indicator, this function's support is \mathbb{R} .

$$g(-1) = 2(-1)^{-3} = -2$$

This violates the condition that

$$g(x) \geq 0 \quad \forall x \in \mathbb{R}.$$

$$\int_{-\infty}^{\infty} g(x) dx \neq 1. \quad \text{Function undefined at } x=0$$

splitting at $x=0$ will give two divergent integrals. \therefore Total integral is undefined.

2 Continuous families

(Continuous) Uniform

DEFINITION

Let $L < R \in \mathbb{R}$. A RV X with PDF

$$f_X(x; L, R) = \frac{1}{R - L} I_{[L, R]}(x),$$

is said to have the $\text{Unif}(L, R)$ distribution.

Continuous Uniform Median

EXERCISE: UNIFORM MEDIAN

For continuous random variables, the **median** is the number m such that

$$\mathbb{P}(X < m) = \mathbb{P}(X > m) = 1/2.$$

Let $X \sim \text{Unif}(L, R)$. What is the median of X ?

$$f_X(x) = \frac{1}{R-L} I_{[L,R]}(x)$$

$$P(X < m) = \int_L^m \frac{1}{R-L} dx$$

$$0.5 = \int_L^m \frac{1}{R-L} dx$$

$$0.5 = \frac{x}{R-L} \Big|_{x=L}^{x=m}$$

$$0.5 = \frac{m-L}{R-L}$$

$$0.5(R-L) = m-L$$

$$m = \frac{R-L}{2} + L = \frac{R+L}{2}$$

Continuous Uniform Median

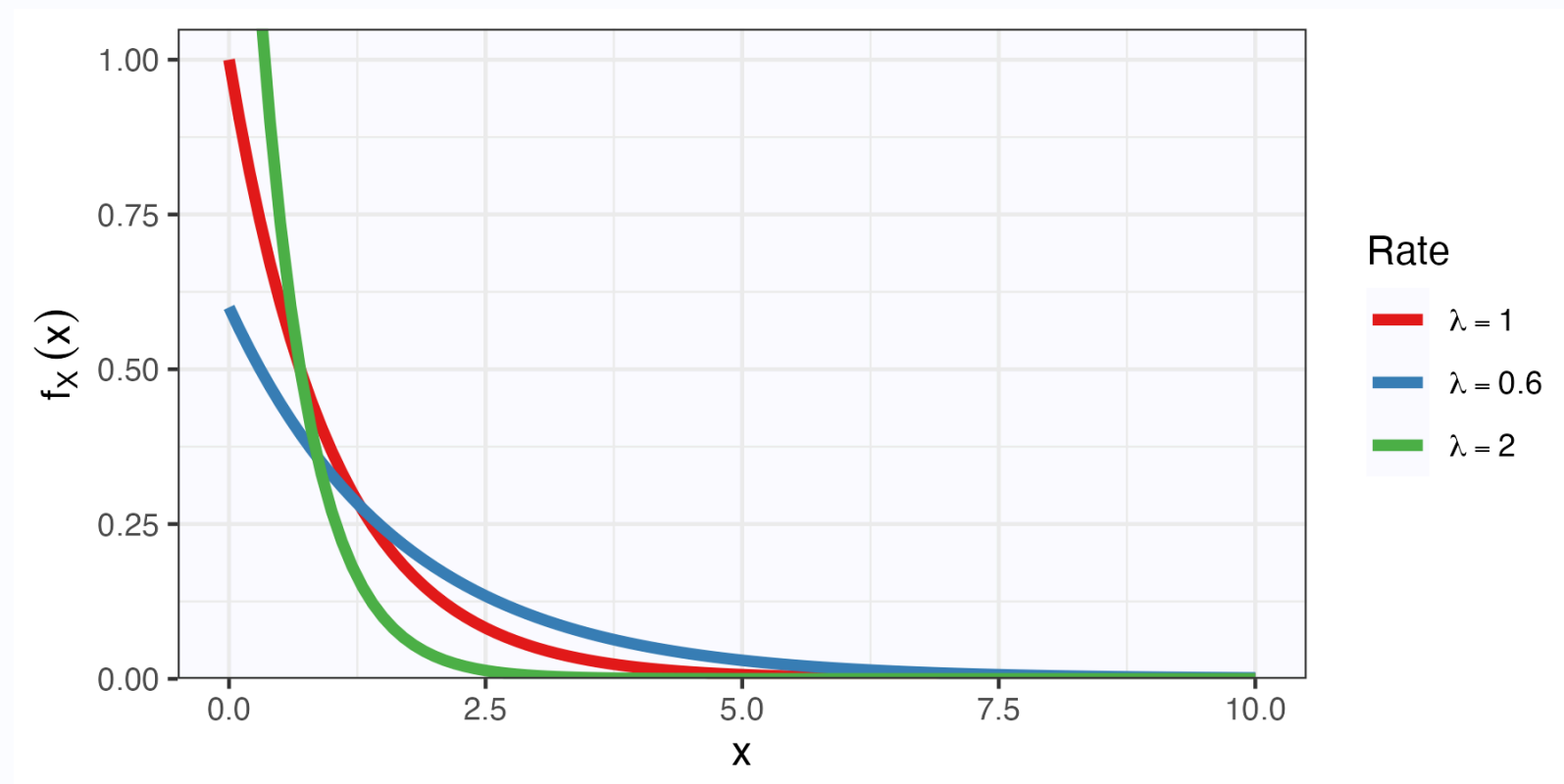
Exponential Distribution

📖 DEFINITION

Let $\lambda > 0$. A RV X with PDF

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x),$$

is said to have the $\text{Exp}(\lambda)$ distribution with rate λ .



Exponential Distribution

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x)$$

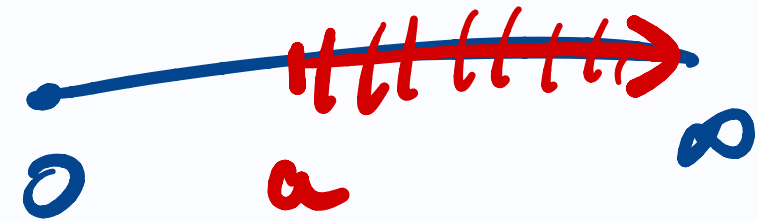
EXERCISE: EXPONENTIAL

Let $Y \sim \text{Exp}(2)$. Find $\mathbb{P}(Y > a)$ for $a > 0$.

$\uparrow \lambda = 2$

$$\begin{aligned} \mathbb{P}(Y > a) &= \int_a^{\infty} 2e^{-2y} dy \\ &= -e^{-2y} \Big|_{y=a}^{y=\infty} \\ &= e^{-2a} \end{aligned}$$

$$f_Y(y; \lambda=2) = 2e^{-2y} I_{[0, \infty)}(y)$$



Aside:

$$\mathbb{P}(Y > \underline{3}) = e^{-2(3)} = e^{-6}$$

(next class!)

Exponential Distribution

Exponential Distribution

$$f_T(t) = \lambda e^{-\lambda t} I_{[0, \infty)}(t)$$

EXERCISE: EXPONENTIAL RATES

A small coffee shop receives customers independently at an average rate of 12 per hour. Let T be the waiting time (in minutes) between consecutive customer arrivals.

- What distribution does T follow, and what is its rate parameter λ ?
- What is the probability that the next customer arrives within 3 minutes?

a) T is time in minutes: $\lambda = \frac{12 \text{ customer}}{1 \text{ hour}} \times \frac{1 \text{ hour}}{60 \text{ min}} = \frac{1}{5}$ ← rate param in min!

$$T \sim \text{Exp}(\lambda = 1/5)$$

b) $P(T \leq 3) = \int_0^3 \frac{1}{5} e^{-1/5 t} dt = 0.4512$

Exponential Distribution

Gamma Distribution

Recall the gamma function:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp(-t) dt, \text{ where } \alpha > 0$$

Useful results:

- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

- If α is an integer, this integral simplifies to $\Gamma(n) = (n - 1)!$

$$\Gamma(6) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- $\Gamma(1/2) = \sqrt{\pi}$

Gamma Distribution

DEFINITION

Let $\alpha, \lambda > 0$. A RV Z with pdf

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z),$$

is said to have the $\text{Gam}(\alpha, \lambda)$ distribution with shape α and rate λ .

Note: $Z \sim \text{Gam}(1, \lambda) \Rightarrow Z \sim \text{Exp}(\lambda)$.

Gamma Distribution

EXERCISE: ~~GAS PUMPS~~ Insurance

The time to process an insurance claim follows $\text{Gam}(\alpha = 2, \lambda = 1/3)$. What is the probability that the claim takes more than 6 hours to process?

integration by parts
 $\int u dv = uv - \int v du$

X : time to process claim

$X \sim \text{Gam}(2, 1/3)$

$$\begin{aligned} P(X \geq 6) &= \int_6^{\infty} \frac{(1/3)^2 x^{2-1} e^{-1/3x}}{\Gamma(2)} dx \\ &= \frac{(1/3)^2}{1} \int_6^{\infty} x e^{-1/3x} dx \\ &= \frac{1}{9} \int_6^{\infty} x e^{-1/3x} dx \end{aligned}$$

$$= \frac{1}{9} \int_6^{\infty} x e^{-1/3x} dx$$

int by parts:

$$u = x$$
$$du = dx$$

$$v = -3e^{-1/3x}$$
$$dv = e^{-1/3x} dx$$

$$= \frac{1}{9} \left[x(-3e^{-1/3x}) \Big|_{x=6}^{x=\infty} - \int_6^{\infty} (-3)e^{-1/3x} dx \right]$$

$$= \frac{1}{9} \left[-3xe^{-1/3x} + 3(-3)e^{-1/3x} \right] \Big|_{x=6}^{x=\infty}$$

$$= \frac{1}{9} \left[0 - (-3(6)e^{-1/3(6)} - 9e^{-1/3(6)}) \right]$$

$$= 3e^{-2} \approx 0.4060$$

$$P(X \geq 6) = 0.4060$$

Kernel and Integration Constant

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z).$$

- PDFs/PMFs must integrate/sum to 1.
- The functional form can be thought of as two pieces:
 1. The “kernel” is the portion that depends on the argument (x or z)
 2. The “normalizing constant” is the part that depends only on parameters; this makes the function integrate to 1.
- The support (given by the indicator function) is part of the kernel.

Kernel and Integration Constant: Example

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z)$$

1. The kernel is $z^\alpha e^{-\lambda z} I_{[0, \infty)}(z)$.
2. The normalizing constant is $\lambda^\alpha / \Gamma(\alpha)$.

We know that

$$1 = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} dz \quad \rightsquigarrow \quad \frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty z^{\alpha-1} e^{-\lambda z} dz.$$

$1 = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty z^{\alpha-1} e^{-\lambda z} dz$ implies

Kernel Matching

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z)$$

EXERCISE: KERNEL MATCHING

1. What is

$$\int_0^\infty z^3 e^{-5z} dz?$$

2. What is

$$\int_0^\infty z \frac{\lambda^4}{\Gamma(4)} z^3 e^{-\lambda z} dz?$$

Hint: Recall that $\Gamma(n) = (n - 1)!$ when $n \in \{1, 2, \dots\}$.

Kernel Matching

$$a) \int_0^{\infty} z^3 e^{-5z} dz = \int_0^{\infty} z^{4-1} e^{-5z} dz$$

looks like $\text{Gam}(\alpha=4, \lambda=5)$

$$= \frac{\Gamma(4)}{5^4} \cdot \frac{5^4}{\Gamma(4)} \int_0^{\infty} z^{4-1} e^{-5z} dz$$

mult. by 1

$$= \frac{\Gamma(4)}{5^4} \underbrace{\int_0^{\infty} \frac{5^4}{\Gamma(4)} z^{4-1} e^{-5z} dz}_{=1}$$

int=1 since Gamma pdf!

$$= \frac{\Gamma(4)}{5^4} \cdot 1$$

$$= \frac{3 \cdot 2 \cdot 1}{5^4} = 0.0096$$

Kernel Matching

$$b) \int_0^{\infty} z \frac{\lambda^4}{\Gamma(4)} z^3 e^{-\lambda z} dz$$

$$= \int_0^{\infty} z^4 \frac{\lambda^4}{\Gamma(4)} e^{-\lambda z} dz$$

$$= \int_0^{\infty} \frac{\lambda^4}{\Gamma(4)} z^{5-1} e^{-\lambda z} dz$$

→ looks like $\text{Gam}(5, \lambda)$

$$= \frac{\Gamma(5)}{\lambda^5} \cdot \frac{\lambda^5}{\Gamma(5)} \cdot \frac{\lambda^4}{\Gamma(4)} \int_0^{\infty} z^{5-1} e^{-\lambda z} dz$$

mult by 1

$$= \frac{\Gamma(5)}{\lambda^5} \cdot \frac{\lambda^4}{\Gamma(4)} \cdot \int_0^{\infty} \frac{\lambda^5}{\Gamma(5)} z^{5-1} e^{-\lambda z} dz$$

pdf of $\text{Gam}(5, \lambda)$

$$= \frac{\Gamma(5)}{\lambda^5} \frac{\lambda^4}{\Gamma(4)} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{\lambda \cdot 3 \cdot 2 \cdot 1} = \frac{4}{\lambda}$$

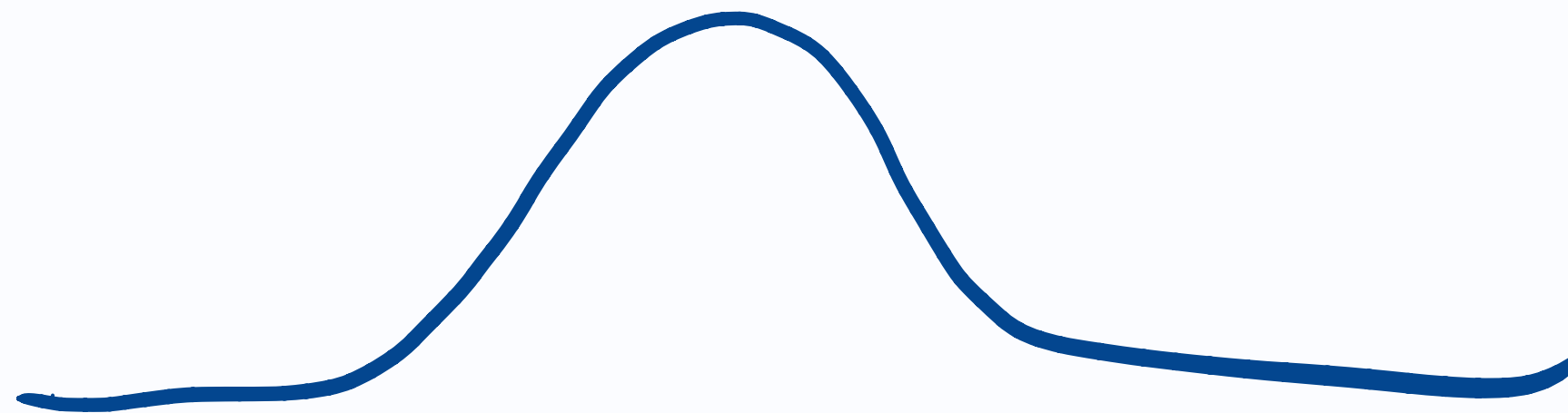
The Normal (Gaussian) distribution

DEFINITION

Let $\mu \in \mathbb{R}$, $\sigma > 0$. A RV Z with pdf

$$f_Z(z; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\},$$

is said to have the $\mathcal{N}(\mu, \sigma^2)$ distribution.



The Normal Distribution

- This distribution is incredibly important.
- The reason is that **it is good for modelling averages**. We'll justify this rigorously later.
- $Z \sim \mathcal{N}(0, 1)$ is called **the standard normal distribution**. When Z is written without context, it is often understood to have this specific distribution.
- Unfortunately

$$\mathbb{P}(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz,$$

does not have a closed form solution.

- Old folks (like me) used tables in textbooks to calculate this (Table D.2 on p. 712 for you).
- Nowadays, we use software.

To do:

- Read [Chapter 2.5](#) before next class
- Assignment 2 due tomorrow, May 27th @ 11:59pm.
- Midterm is next Tuesday during class.