Lecture 8

Cumulative Distribution Functions and Transformations

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a cumulative distribution function (CDF)
- Calculate CDFs from PDF/PMFs
- Perform change of variables to calculate CDFs and PDFs/PMFs for functions of random variables.

1 Cumulative Distribution Functions (CDFs)

Cumulative Distribution Functions (CDFs)

📖 **DEFINITION**

The cumulative distribution function of a random variable X is

$$F_X(x) = \mathbb{P}(X \leq x), \qquad \text{for } x \in \mathbb{R}$$

📐 **THEOREM**

Let X be a random variable with CDF F_X. Then:

1. $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$;
2. $F(x) \leq F(y)$ for all $x \leq y$;
3. $\lim_{a \to -\infty} F_X(a) = 0$,
4. $\lim_{a \to +\infty} F_X(a) = 1$

CDFs for Discrete and Continuous RVs

Discrete RV

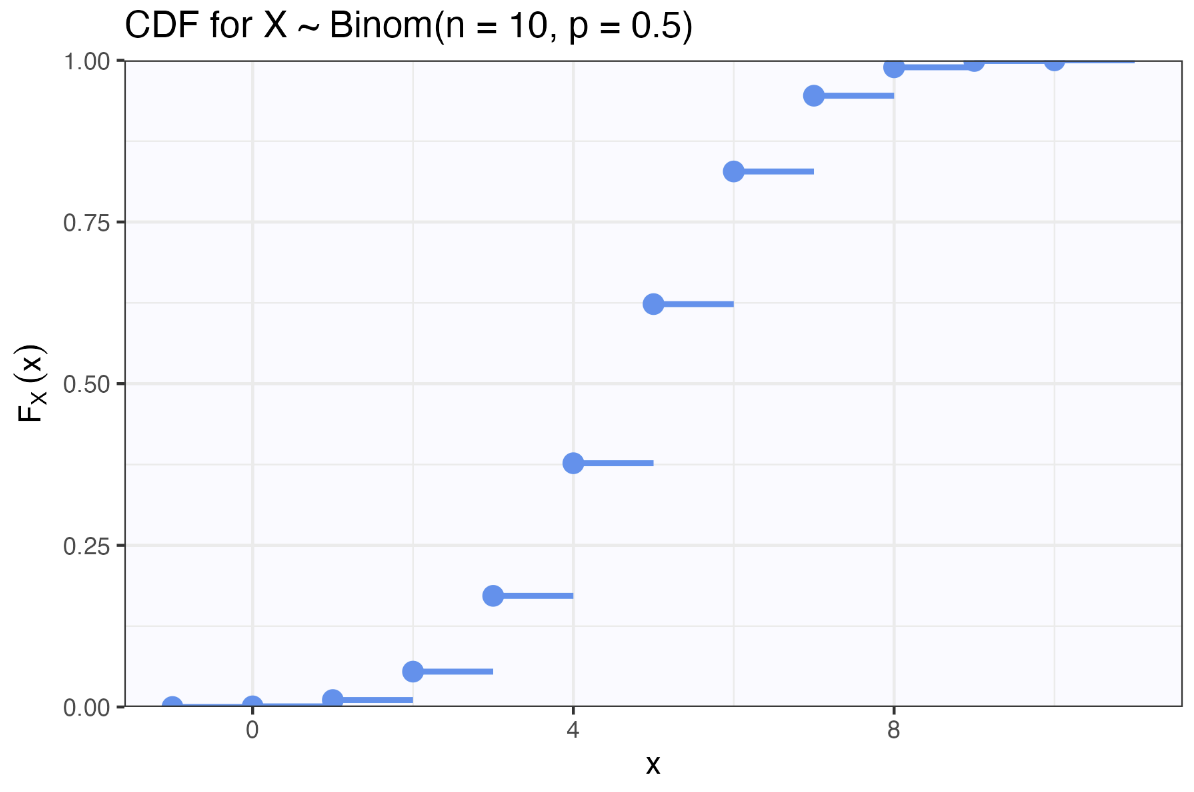
$$F_X(x) = \sum_{t \leq x} p_X(t)$$

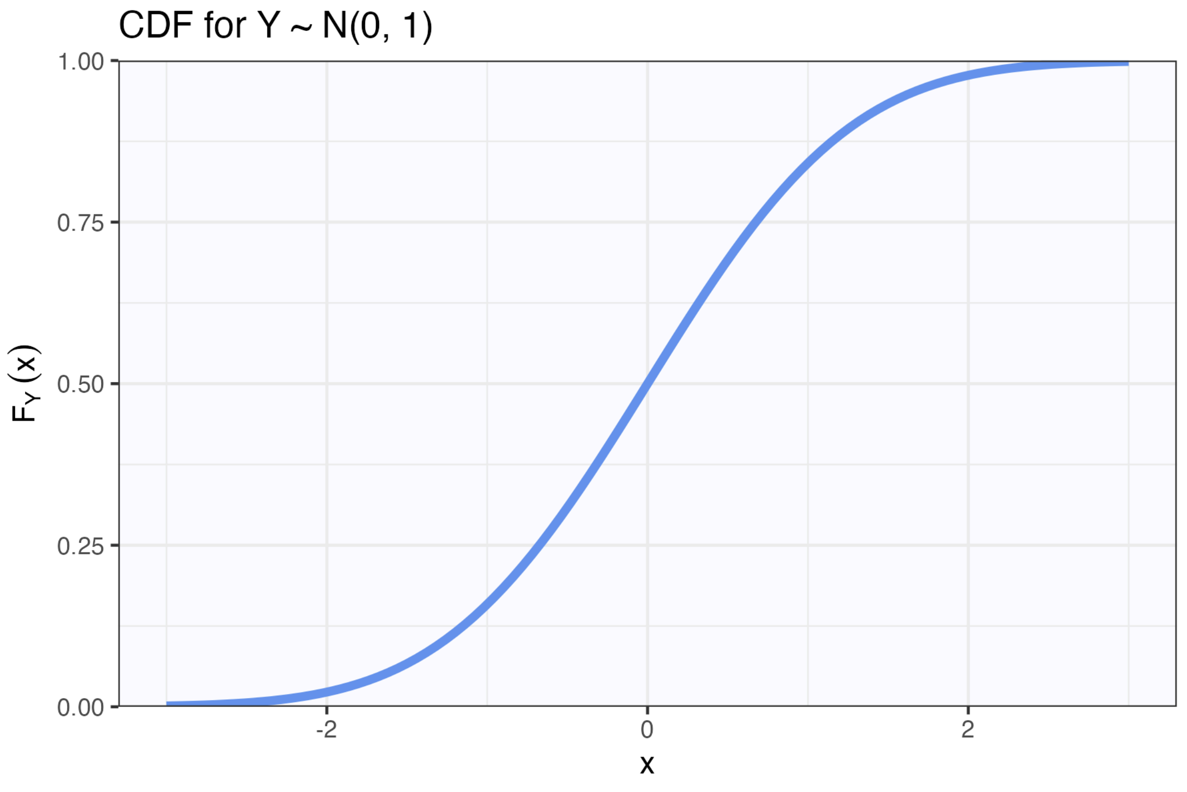
- The CDF is a step function and right-continuous.

Continuous RV

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t$$

- The CDF is continuous (left and right limits are equal).





Continuous CDF Example

Let $X \sim \mathrm{Exp}(\lambda)$, with pdf

$$f_X(x) = \lambda e^{-\lambda x} I_{[0,\infty)}(x).$$

💡 **EXAMPLE**

Find $F_X\,(x)$. Use the CDF to calculate $\mathbb{P}(x \leq 10)$ when $\lambda = 1/5$.

Continuous CDF Example

Discrete CDF Example

💡 EXAMPLE

Consider rolling one fair six-sided die, so that $\Omega = \{1, 2, 3, 4, 5, 6\}$, with $\mathbb{P}(\omega) = 1/6$ for each $\omega \in \Omega$ Let X be the number showing on the die. What is $F_X(x)$? Use this to find $\mathbb{P}(X \leq 5)$.

CDFs and Distributions

📐 THEOREM

Let X be any random variable with CDF F_X. Let B be any subset of $\mathbb{R}$. Then $\mathbb{P}(X \in B)$ can be determined solely from F_X.

- The CDF contains all the information about the distribution of X.
- It doesn't matter whether X is discrete, continuous, or anything else.

📌 COROLLARY

Let X be an absolutely continuous random variable with cumulative distribution function F_X. Let

$$f_X(x) = \frac{d}{dx} F_X(x) = F'(x).$$

Then, $f(x)$ is the density function for X.

Aside: The discrete case involves finding the differences between consecutive CDF values.

Mixture Distributions

PROPOSITION

Let $X_1, X_2, \ldots$ be a random variables with CDFs $F_{X_1}, F_{X_2}, \ldots$.

For any constants p_i such that $p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$, $F_G(x) = \sum_{i=1}^{k} p_i F_{X_i}(x)$ is the CDF of the **mixture** of F_{X_i}.

Mixture Distributions

💡 **EXAMPLE**

Suppose a bag contains two coins. You choose one at random and flip it once. Let X be the number of heads.

Coin A is a fair coin, with $\mathbb{P}(\text{flip Heads}) = 0.5$ Coin B is a loaded coin, with $\mathbb{P}(\text{flip Heads}) = 0.9$

Write out the CDF for the number of heads flipped.

Mixture Distributions

Mixture Distributions

Consider the scores of a midterm. Suppose that there are three types of students, modeled as follows:

1. Poorly prepared students who did not study much (30%): $X_1 \sim \mathcal{N}(\mu = 50, \sigma = 10)$.
2. Well prepared students who studied a lot (60%): $X_2 \sim \mathcal{N}(\mu = 80, \sigma = 8)$.
3. Students who didn't take the exam at all (10%): $X_3 = 0$ with probability 1.

✏️ **EXERCISE: MIXTURE DISTRIBUTION FOR MIDTERM**

Find the CDF of the overall score distribution X. Write your answer in terms of the standard normal CDF $\Phi(\cdot)$.

Hint: If $Y \sim \mathcal{N}(\mu, \sigma)$, then $F_Y(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$.

Mixture Distributions

2 Transformations of CDFs

One-Dimensional Change of Variables

Sometimes we want to perform transformations on a random variable. This is when we can turn to **change of variables**.

- Let X be a random variable with some distribution
- Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$.
- We want to find the distribution of Y.

There are two ways to do this: the **distribution method** and the **Jacobian method**. We will present both.

One-Dimensional Change of Variables: Distribution Method

The **distribution method** just uses the definition:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}).$$

If we can characterize these sets, we can find the distribution of Y. This method always works, and is easy for discrete random variables.

THEOREM

Let X be a discrete random variable with PMF $p_X(x)$. Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$. Then the PMF of Y is given by

$$p_Y(y) = \sum_{x : g(x) = y} p_X(x) = \sum_{x \in g^{-1}(y)} p_X(x).$$

One-Dimensional Change of Variables: Distribution Method

Let's make this method a little more algorithmic.

If you want to find the distribution of $Y = g(X)$, then you can do the following:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}) = \cdots$$

1. Start with the definition of the distribution of Y.
2. Use the definition of Y in terms of X to rewrite the probability in terms of X.
3. Use the distribution of X to compute the probability that $g(X) \in A$.
4. Then play algebra games to get it into a nice form; possibly "match" a known PDF or CDF.

One-Dimensional Change of Variables: Distribution Method (Discrete)

Let's start with a straightforward example:

💡 EXAMPLE

- Let $X \sim \text{Binom}(n, \theta)$, for some $\theta \in (0, 1)$.
- Let $Y = n - X$.

Find the PMF of Y.

One-Dimensional Change of Variables: Distribution Method (Discrete)

One-Dimensional Change of Variables: Distribution Method (Continuous)

💡 **EXAMPLE**

Let $X \sim \mathrm{Exp}(\lambda)$, find the distribution of $Y = 3X$.

One-Dimensional Change of Variables: Distribution Method (Continuous)

💡 **EXAMPLE**

If $X \sim \text{Exp}(\lambda)$, what is the CDF of $Y = X + 5$?

One-Dimensional Change of Variables: Useful Results

The following useful results follow from the previous examples

📚 **PROPOSITION**

If X be a random variables with CDFs F_X Then the the following hold:

1. The RV $Y = X + c$ has CDF $F_X(x - c)$;
2. The RV $Y = kX$ has CDF $F_X(x/k)$ for any $k > 0$;

One-Dimensional Change of Variables: Jacobian Method

For continuous random variables, you can also use the distribution method, and sometimes this is the easiest way. The other common method is the **Jacobian method**.

THEOREM

Let X be an (absolutely) continuous random variable, with density function f_X. Let $Y = h(X)$, where $h : \mathbb{R} \to \mathbb{R}$ is a function that is differentiable and monotonic.

Then Y is also absolutely continuous, and its density function f_Y is given by

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(h^{-1}(y)) \right|,$$

where $h^{-1}(y)$ is the unique number x such that $h(x) = y$.

One-Dimensional Change of Variables: Jacobian Method

💡 **EXAMPLE**

- Let $X \sim \mathrm{Unif}(0, 1)$.
- Let $Y = -\log(X)$.

Find the PDF of Y.

One-Dimensional Change of Variables: Jacobian Method

✏️ **EXERCISE:**

1. Let $X \sim \text{Unif}\,(-1, 1)$. Use the distribution method to find the PDF of $Z = X^2$.
2. Let $X \sim \text{Gam}(\alpha, \lambda)$. Use the Jacobian method to find the PDF of $Y = 1/X$.

Recall the Gamma PDF:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

One-Dimensional Change of Variables: Jacobian Method

One-Dimensional Change of Variables: Jacobian Method

This is the end of the "testable" content for your midterm.

- Your midterm will cover materials from Lectures 1 - 8
- DATE: in class on Tuesday June 2
- LENGTH: 1 hour and 50 minutes in length.
- You may bring in one (1) "cheat sheet":
  - Must be HAND WRITTEN with pen/pencil on said sheet of paper (not typed, not photo copied, not printed, not written on an iPad)
  - Must be on 8.5 by 11 inch sheet of paper or smaller
  - You may write on both sides
  - No magnifying glasses or anything else silly
  - **I will confiscate cheatsheets that do not follow these rules** 🥀
  - I do not care what is written on it
- Exam is hand written on paper, bring something to write with
- You may bring a non-programmable, non-graphing calculator.

To do:

- Read Chapter 2.7 before next class
- Assignment 2 due tonight, May 27th @ 11:59pm.
- Start reviewing for your midterm. No assignment due next week.