

Lecture 9

Continuous Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a probability density function
- Use the continuous PDFs to solve for probabilities

1 Continuous Random Variables

Continuous Random Variables

DEFINITION

A RV X is **continuous** if

$$\mathbb{P}(X = x) = 0,$$

for every $x \in \mathbb{R}$.

- This is the most rigorous way to define continuous RVs, but it doesn't provide much intuition.
- We need a bit more to make our intuition match the math.

Absolutely Continuous RVs and Density Functions

DEFINITION

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called a **density function** if $f(x) \geq 0$, $\forall x \in \mathbb{R}$ and $\int f(x)dx = 1$.
1) 2)

DEFINITION

A RV X is absolutely continuous if there exists a density function f such that

$$\mathbb{P}(\underline{a} \leq X \leq \underline{b}) = \int_a^{\underline{b}} f(x)dx$$

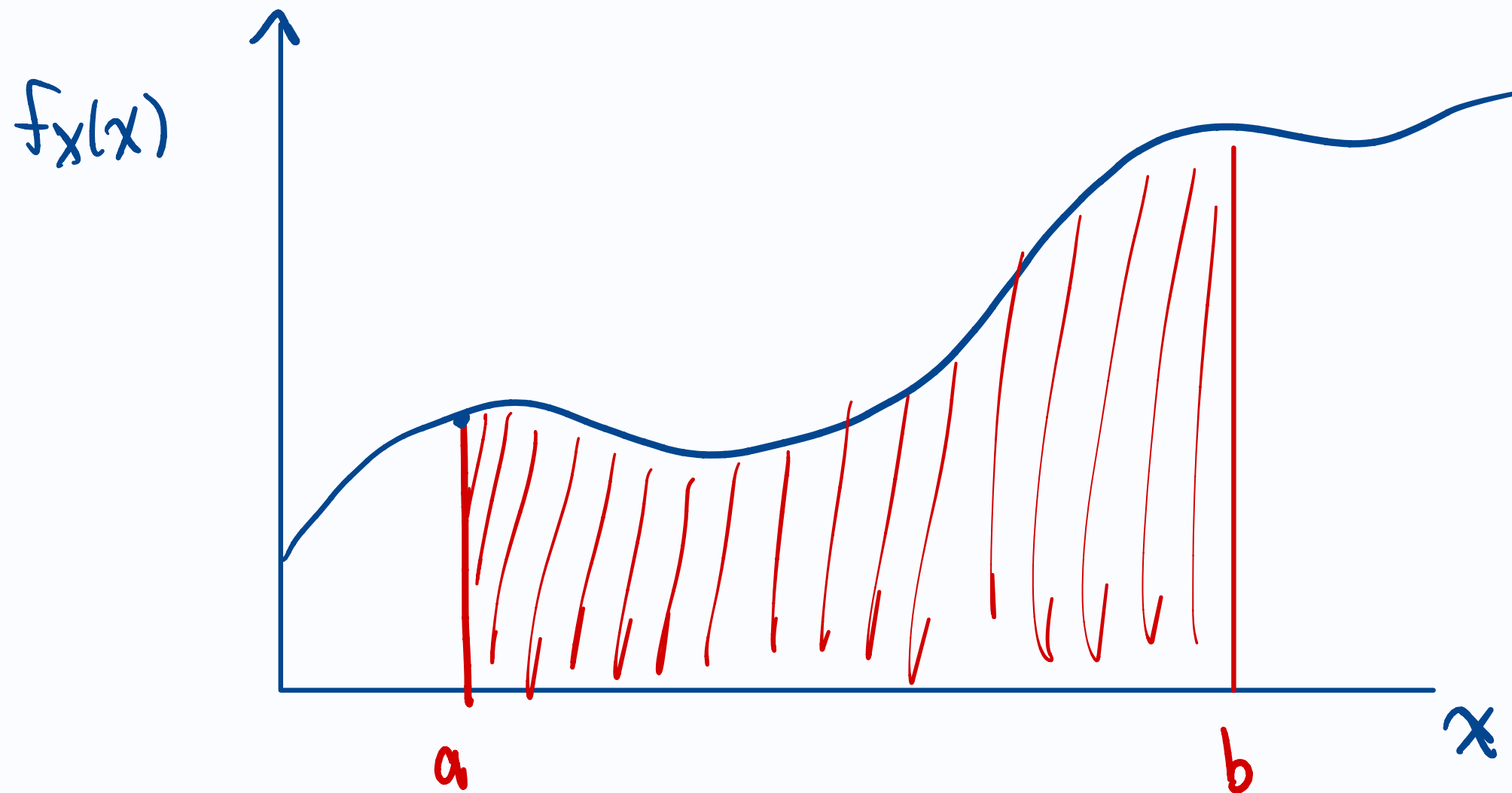
whenever $a \leq b$.

We call such a density function a **probability density function** (PDF) and will often use the notation $f_X(x)$ to denote its relationship to X .

→ in this class, we will only see absolutely continuous densities!

Probability of $a \leq X \leq b$ visually

$P(a \leq X \leq b) = P(a < X < b)$ because $P(X=a)=0$ by def'n since continuous!



$$= \int_a^b f_X(x) dx$$

Absolutely Continuous RVs

THEOREM

Let X be an absolutely continuous random variable. Then X is a continuous random variable (i.e., $\mathbb{P}(X = a) = 0$ for all $a \in \mathbb{R}$)

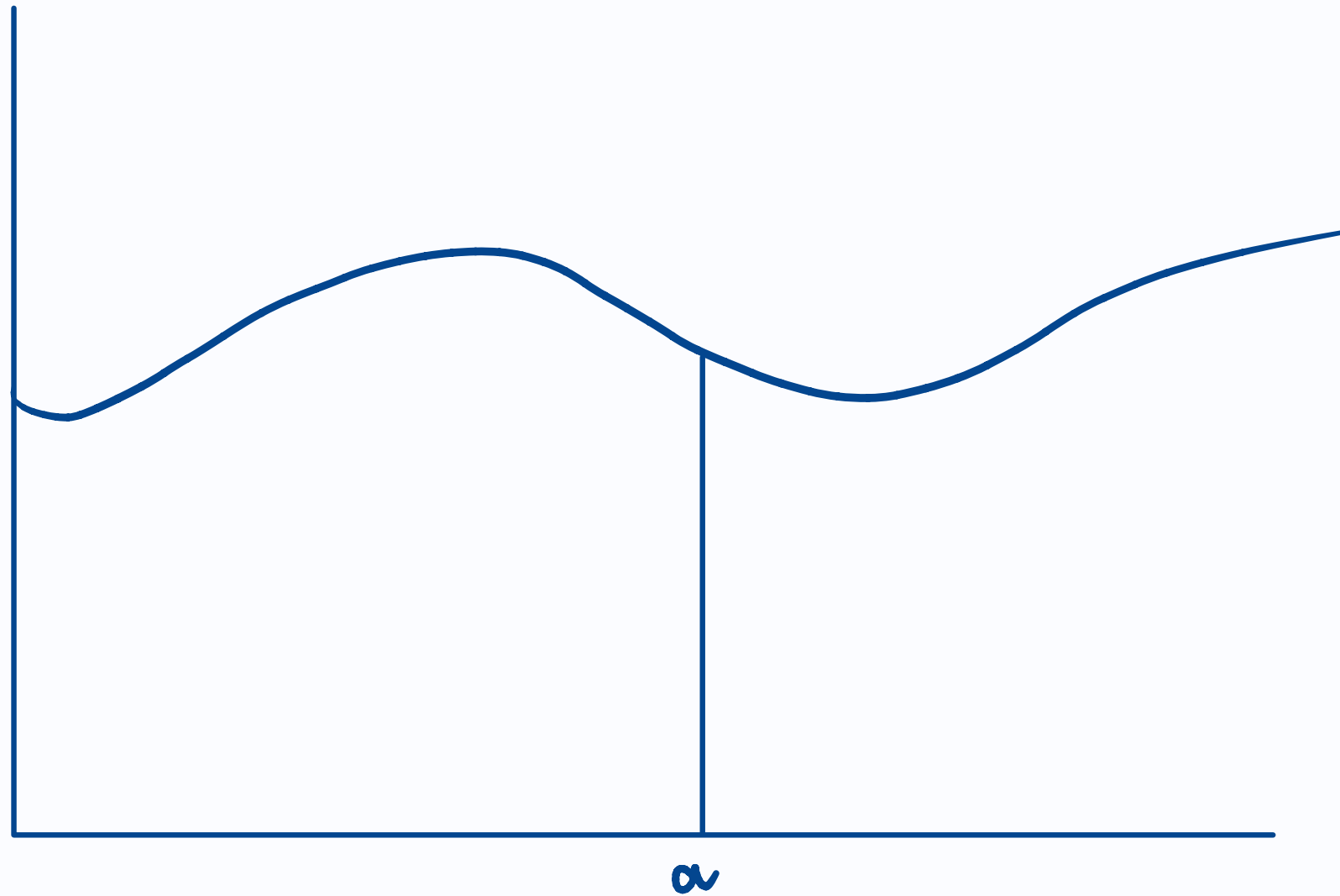
 PROOF $= \mathbb{P}(a \leq X < a) =$

$$\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)dx = 0$$

Note

Note: the converse is not necessarily true. That is, not all continuous distributions you will come across in statistics (in general) are **absolutely** continuous. We will stick to **absolutely continuous distributions** in this course.

Probability of $X = a$ visually



→ no area!
 $P(X=a) = 0$

Comparison to Discrete RVs

- Consider some set $A \subset \mathbb{R}$.
- Let X be a random variable.

Probability of A for discrete RV:

$$\mathbb{P}(X \in A) = \sum_{x \in A} p_X(x).$$

↑ PMF

Probability of A for an (absolutely) continuous RV:

$$\mathbb{P}(X \in A) = \int_A f_X(x) dx.$$

↑ PDF

Continuous PDFs

💡 EXAMPLE

Let X be a RV with PDF

$$f_X(x) = 2x^{-3}I_{[1,\infty)}(x).$$

- Is f a valid density function?
- What is $\mathbb{P}(X = 2)$?
- What is $\mathbb{P}(0 < X < 2)$?
- Suppose we defined $g(x) = 2x^{-3}$ (no indicator function). Is $g(x)$ a density function? Does it have the same support?

a) $f_X(x) \geq 0 \quad \forall x \in \mathbb{R} : \quad \frac{2}{x^3}$ for $x \in [1, \infty)$ will never be negative and 0 otherwise ✓

$$\int_{\mathbb{R}} f_X(x) dx = \int_1^{\infty} 2x^{-3} dx = \frac{2x^{-2}}{-2} \Big|_{x=1}^{x=\infty} = 0 - (-1) = 1 \quad \checkmark$$

$\therefore$ valid density!

Continuous PDFs

$$f_X(x) = 2x^{-3} I_{[1, \infty)}(x)$$

b) $P(X=2) = 0$ because X is absolutely continuous!

aside: $P(2 \leq X < 2) = \int_2^2 2x^{-3} dx = 0$

$$\begin{aligned} \text{c) } P(0 < X < 2) &= \int_0^2 f_X(x) dx = \int_0^2 2x^{-3} I_{[1, \infty)}(x) dx \\ &= \int_1^2 2x^{-3} dx = 3/4 \end{aligned}$$

d) $g(x) = 2x^{-3}$: $g(x) < 0$ for $x = -1$ (and others) $\times$

$$\int g(x) dx = \int_{-\infty}^{\infty} 2x^{-3} dx \neq 1$$

This function is undefined at $x=0$. Splitting the integral at $x=0$ gives two divergent integrals. The total integral = undefined.

Continuous PDFs

EXERCISE: EXAMPLE CONTINUOUS RV

Let $f_X(x) = \frac{1}{5} I_{[0,5]}(x)$.

a. What is $\mathbb{P}(X \leq 3)$?

b. What is $\mathbb{P}(X < 3)$?

c. What is $\mathbb{P}(-4 < X < 10)$?

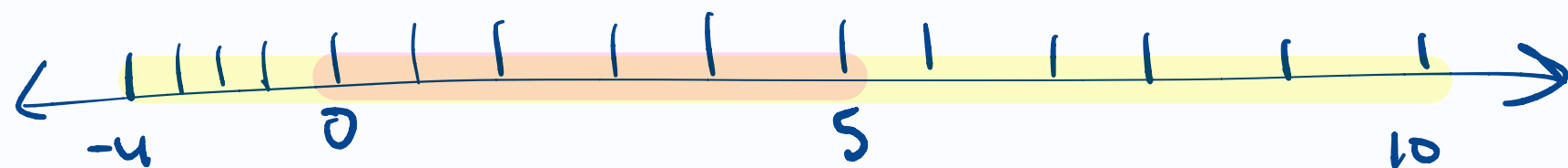
d. What is $\mathbb{P}(X \geq 2)$?

$$\begin{aligned} \text{a/b)} \quad \mathbb{P}(X \leq 3) &= \mathbb{P}(X < 3) = \int_{-\infty}^3 \frac{1}{5} I_{[0,5]}(x) dx \\ &= \int_0^3 \frac{1}{5} dx = \frac{1}{5} x \Big|_0^3 = \frac{3}{5} \end{aligned}$$

Continuous PDFs

$$f_X(x) = \frac{1}{5} \mathbb{I}_{[0,5]}(x)$$

$$c) P(-4 < X < 10) = \int_0^5 \frac{1}{5} dx = \frac{1}{5} x \Big|_{x=0}^{x=5} = 1$$



$$d) P(X \geq 2) = \int_2^5 \frac{1}{5} dx = \frac{1}{5} x \Big|_{x=2}^{x=5} = \left(1 - \frac{2}{5}\right) = \frac{3}{5} \quad \leftarrow$$

OR

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \int_0^2 \frac{1}{5} dx = 1 - \frac{1}{5} x \Big|_{x=0}^{x=2} = \left(1 - \left(\frac{2}{5}\right)\right) = \frac{3}{5}$$

Reminders

- No class on Monday (thanksgiving)
- Midterm 1 on Friday October 16th (all material up to today's class) *(and including)*
- Start prepping your cheat sheet
 - MUST BE HAND WRITTEN! No photo copies, typed, or tablet-written and printed sheets allowed
 - 8.5 inches x 11 inches or less
 - You can have one single page, and can write on both sides

Checkpoint 3

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing calculator
- Writing materials

Phones and laptops put away, please! You can have 15 minutes.