

Lecture 11

Continuous Distributions (Part 2)

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Last modified — 18 Oct 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Identify various continuous families or RV
- Use known distributions to solve for word problems

1 Continuous families

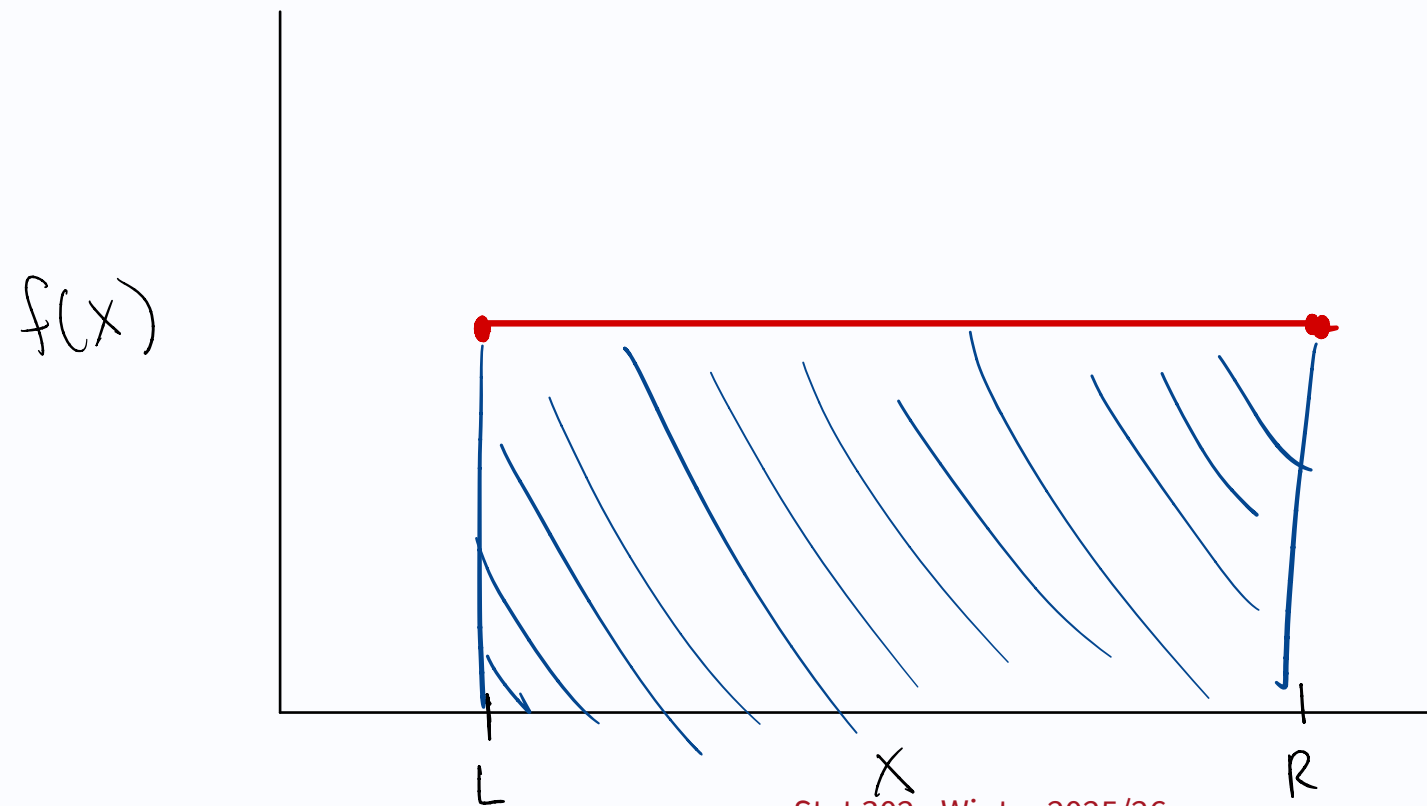
(Continuous) Uniform

📖 DEFINITION

Let $L < R \in \mathbb{R}$. A RV X with PDF

$$f_X(x; L, R) = \frac{1}{R - L} I_{[L, R]}(x),$$

is said to have the $\text{Unif}(L, R)$ distribution.



Continuous Uniform Median

$$P(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

EXERCISE: UNIFORM MEDIAN

For continuous random variables, the **median** is the number m such that

$$\mathbb{P}(X < m) = \mathbb{P}(X > m) = 1/2.$$

Let $X \sim \text{Unif}(L, R)$. What is the median of X ? $\rightarrow f_X(x) = \frac{1}{R-L} I_{[L, R]}(x)$

$$P(X < m) = \int_{-\infty}^m f_X(x) dx$$

$$0.5 = \int_L^m \frac{1}{R-L} dx$$

$$0.5 = \frac{1}{R-L} (x) \Big|_{x=L}^{x=m}$$

$$\begin{aligned} \Rightarrow 0.5 &= \frac{m-L}{R-L} \\ \Rightarrow R-L(0.5) + L &= m \\ \frac{R-L}{2} + L &= m \\ \frac{R+L}{2} &= m \end{aligned}$$

(this is halfway between L and R)

Continuous Uniform Median

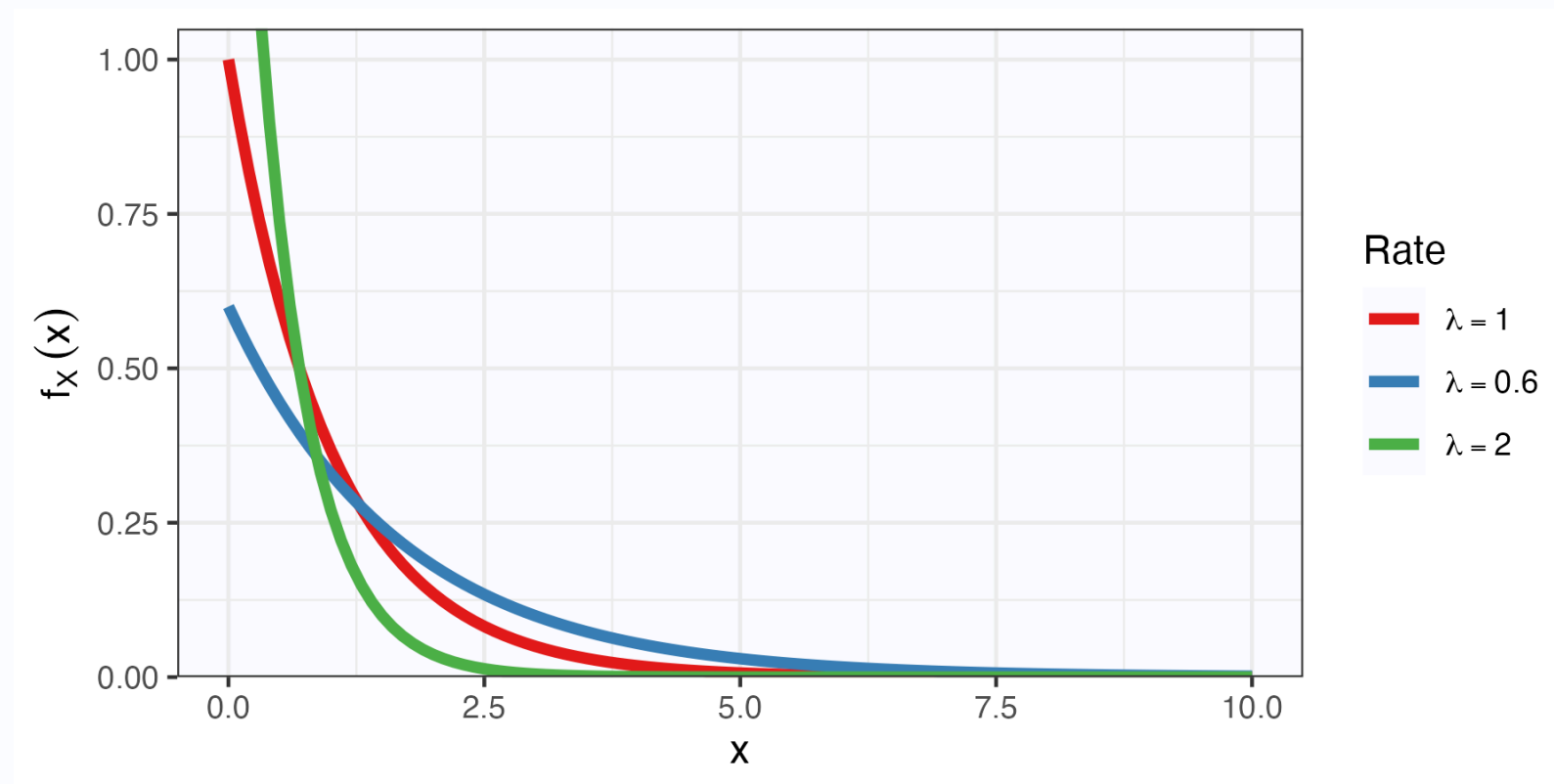
Exponential Distribution

📖 DEFINITION

Let $\lambda > 0$. A RV X with PDF

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x),$$

is said to have the $\text{Exp}(\lambda)$ distribution with rate λ .



Exponential Distribution

- The exponential distribution is good for modelling the ~~number of~~ time between independent, random events (i.e., wait or inter-arrival times).
- It is related to the Poisson model, which models the number of events within a given interval!

Exponential Distribution

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x)$$

EXERCISE: EXPONENTIAL

Let $Y \sim \text{Exp}(2)$. Find $\mathbb{P}(Y > a)$ for $a > 0$.

↑

$\lambda = 2$

$$\begin{aligned} P(Y > a) &= \int_a^{\infty} 2e^{-2y} dy \\ &= \left. \frac{2e^{-2y}}{-2} \right|_{y=a}^{y=\infty} \\ &= 0 - (-e^{-2a}) = e^{-2a} \end{aligned}$$

$$P(Y > 3) = e^{-2(3)} = e^{-6}$$

$$P(Y > 100) = e^{-2(100)} = e^{-200}$$

more on this
next class
Ü

Exponential Distribution

Exponential Distribution

$$f_X(x) = \lambda e^{-\lambda x} I_{[0, \infty)}(x)$$

EXERCISE: EXPONENTIAL RATES

A small coffee shop receives customers independently at an average rate of 12 per hour. Let T be the waiting time (in minutes) between consecutive customer arrivals.

- What distribution does T follow, and what is its rate parameter λ ?
- What is the probability that the next customer arrives within 3 minutes?

a) T is a time between independent events $\rightarrow$ Exponential.

$$\lambda = 12 \frac{\text{cust}}{\text{hour}} \times \frac{1 \text{ hour}}{60 \text{ min}} = \frac{1}{5} \frac{\text{customers}}{\text{min}} \quad T \sim \text{Exp}(\frac{1}{5})$$

$$b) P(T \leq 3) = \int_0^3 \frac{1}{5} e^{-1/5 t} dt = 0.4512$$

Exponential Distribution

Gamma Distribution

Recall the gamma function:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp(-t) dt, \text{ where } \alpha > 0$$

Useful results:

- $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$
- If α is an integer, this integral simplifies to $\Gamma(n) = (n - 1)!$
- $\Gamma(1/2) = \sqrt{\pi}$

$$\Gamma(4) = 3 \times 2 \times 1$$

Gamma Distribution

DEFINITION

Let $\alpha, \lambda > 0$. A RV Z with pdf

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z),$$

is said to have the $\text{Gam}(\alpha, \lambda)$ distribution with shape α and rate λ .

Note: $Z \sim \text{Gam}(1, \lambda) \Rightarrow Z \sim \text{Exp}(\lambda)$.

Gamma Distribution

EXERCISE: INSURANCE CLAIMS

The time to process an insurance claim follows $\text{Gam}(\alpha = 2, \lambda = 1/3)$. What is the probability that the claim takes more than 6 hours to process?

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{[0, \infty)}(x)$$

$$P(X \geq 6) = \int_6^\infty \frac{(1/3)^2}{\Gamma(2)} x^{2-1} e^{-1/3 x} dx$$

integration by parts
 $\int u dv = uv - \int v du$

Gamma Distribution

$$\Gamma(2) = (2-1)! = 1! = 1$$

$$P(X \geq 6) = \int_6^{\infty} \frac{(1/3)^2}{\Gamma(2)} x^{2-1} e^{-1/3 x} dx$$

$$= \frac{1}{9} \left[\int_6^{\infty} x e^{-1/3 x} dx \right]$$

$$= \frac{1}{9} \left[x(-3e^{-1/3 x}) \Big|_{x=6}^{x=\infty} - \int_6^{\infty} (-3)e^{-1/3 x} dx \right]$$

$$= \frac{1}{9} \left[-3x e^{-1/3 x} \Big|_{x=6}^{x=\infty} - 3(3)e^{-1/3 x} \Big|_{x=6}^{x=\infty} \right]$$

$$= \frac{1}{9} \left[(0+0) - \left((-3(6)e^{-1/3(6)}) - 9e^{-1/3(6)} \right) \right]$$

$$= 3e^{-2} = 0.4060$$

$$\begin{aligned} u &= x \\ du &= dx \end{aligned} \qquad \begin{aligned} v &= -3e^{-1/3 x} \\ dv &= e^{-1/3 x} dx \end{aligned}$$

Kernel and Integration Constant

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z).$$

- PDFs/PMFs must integrate/sum to 1.
- The functional form can be thought of as two pieces:
 1. The “kernel” is the portion that depends on the argument (x or z)
 2. The “normalizing constant” is the part that depends only on parameters; this makes the function integrate to 1.
- The support (given by the indicator function) is part of the kernel.

Kernel and Integration Constant: Example

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z)$$

1. The kernel is $z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z)$.
2. The normalizing constant is $\lambda^\alpha / \Gamma(\alpha)$.

We know that

$$1 = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} dz$$

which implies

$$\frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty z^{\alpha-1} e^{-\lambda z} dz.$$

Kernel Matching

PDF of
 $\text{Gam}(\alpha, \lambda)$

$$f_Z(z; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z)$$

EXERCISE: KERNEL MATCHING

1. What is

$$\int_0^\infty z^3 e^{-5z} dz?$$

2. What is

$$\int_0^\infty z \frac{\lambda^4}{\Gamma(4)} z^3 e^{-\lambda z} dz?$$

→ simplify what you can

Hint: Recall that $\Gamma(n) = (n-1)!$ when $n \in \{1, 2, \dots\}$.

Kernel Matching

$$a) \int_0^{\infty} z^3 e^{-5z} dz = \int_0^{\infty} z^{4-1} e^{-5z} dz$$

$$= \underbrace{\frac{\Gamma(4)}{5^4} \frac{5^4}{\Gamma(4)}}_{\text{mult. by 1}} \int_0^{\infty} z^{4-1} e^{-5z} dz$$

$$= \frac{\Gamma(4)}{5^4} \underbrace{\int_0^{\infty} \frac{5^4}{\Gamma(4)} z^{4-1} e^{-5z} dz}_{\text{since Gamma PDF}}$$

$$= \frac{\Gamma(4)}{5^4}$$

$$= \frac{(4-1)!}{5^4} = 0.0096$$

looks like $\text{Gam}(d=4, \lambda=5)$
 $\frac{\lambda^d}{\Gamma(d)} \int_0^{\infty} z^{d-1} e^{-\lambda z} dz$

Kernel Matching

$$b) \int_0^{\infty} z \frac{\lambda^4}{\Gamma(4)} z^3 e^{-\lambda z} dz = \frac{\lambda^4}{\Gamma(4)} \int_0^{\infty} z^4 e^{-\lambda z} dz$$

$$= \frac{\lambda^4}{\Gamma(4)} \int_0^{\infty} \underbrace{z^{5-1}} e^{-\lambda z} dz$$

looks like $\text{Gam}(d=5, \lambda)$

$$= \frac{\lambda^4}{\Gamma(4)} \left(\frac{\Gamma(5)}{\lambda^5} \right) \underbrace{\int_0^{\infty} \frac{\lambda^5}{\Gamma(5)} z^{5-1} e^{-\lambda z} dz}_{=1}$$

= 1 PDF of $\text{Gam}(5, \lambda)$

$$= \frac{\lambda^4}{\Gamma(4)} \frac{\Gamma(5)}{\lambda^5} = \frac{4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \cdot \lambda} = \frac{4}{\lambda}$$

The Normal (Gaussian) distribution

DEFINITION

Let $\mu \in \mathbb{R}$, $\sigma > 0$. A RV Z with pdf

$$f_Z(z; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\},$$

is said to have the $\mathcal{N}(\mu, \sigma^2)$ distribution.

The Normal Distribution

- This distribution is incredibly important.
- The reason is that it is good for modelling averages. We'll justify this rigorously later.
- $Z \sim \mathcal{N}(0, 1)$ is called the standard normal distribution. When Z is written without context, it is often understood to have this specific distribution.
- Unfortunately

$$\mathbb{P}(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z,$$

does not have a closed form solution.

- Can use software or tables in textbooks to calculate this (Table D.2 on p. 712 for you).

Upcoming Assessments:

- No checkpoints this week - relax after the midterm!
- **Checkpoint 4 (Sections 2.5 - 2.6: Continuous RV and CDFs) is on Friday Oct 30th**