

Lecture 12

Cumulative Distribution Functions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define a CDF
- Use a CDF to calculate cumulative probabilities
- Convert between PDFs and CDFs
- Find Mixture Distributions

1 Cumulative Distribution Functions (CDFs)

Cumulative Distribution Functions (CDFs)

DEFINITION

The cumulative distribution function of a random variable X is

$$F_X(x) = \mathbb{P}(X \leq x), \quad \text{for } x \in \mathbb{R}$$

THEOREM

Let X be a random variable with CDF F_X . Then:

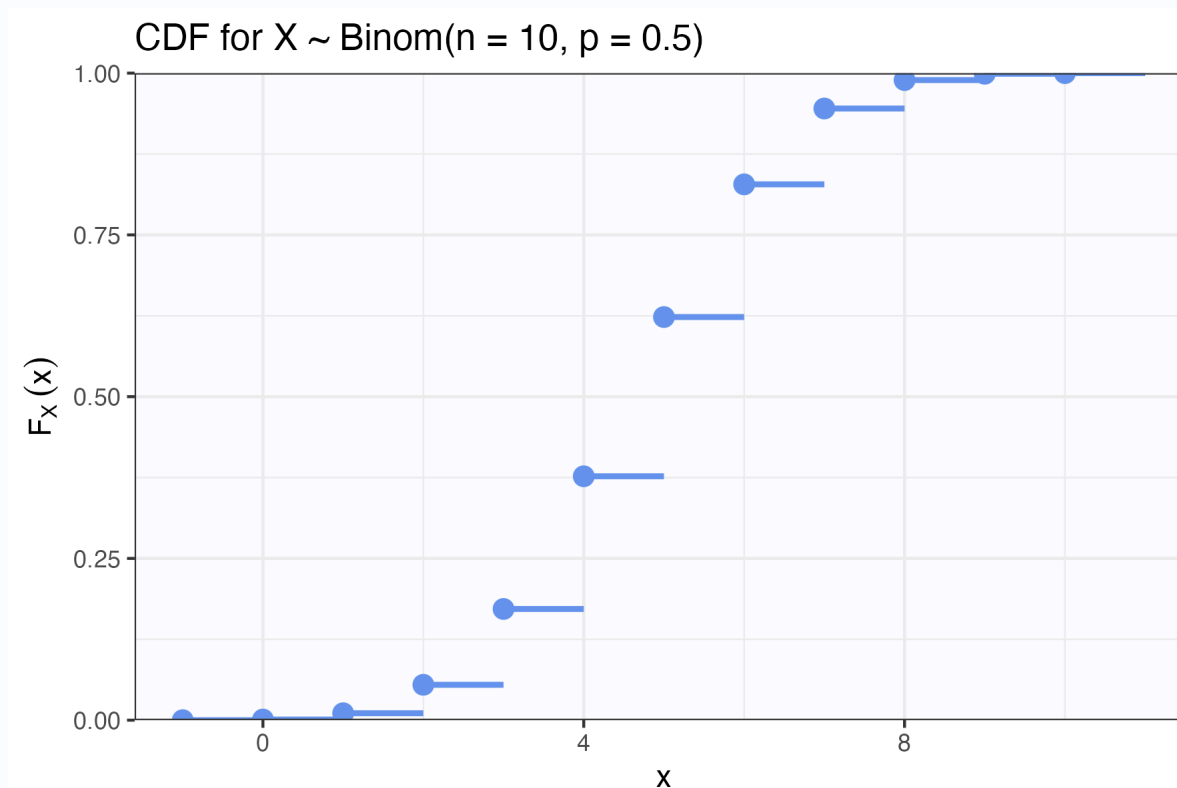
1. $0 \leq F_X(x) \leq 1$ for all $x \in \mathbb{R}$;
2. $F(x) \leq F(y)$ for all $x \leq y$;
3. $\lim_{a \rightarrow -\infty} F_X(a) = 0$,
4. $\lim_{a \rightarrow +\infty} F_X(a) = 1$

CDFs for Discrete and Continuous RVs

Discrete RV

$$F_X(x) = \sum_{t \leq x} p_X(t)$$

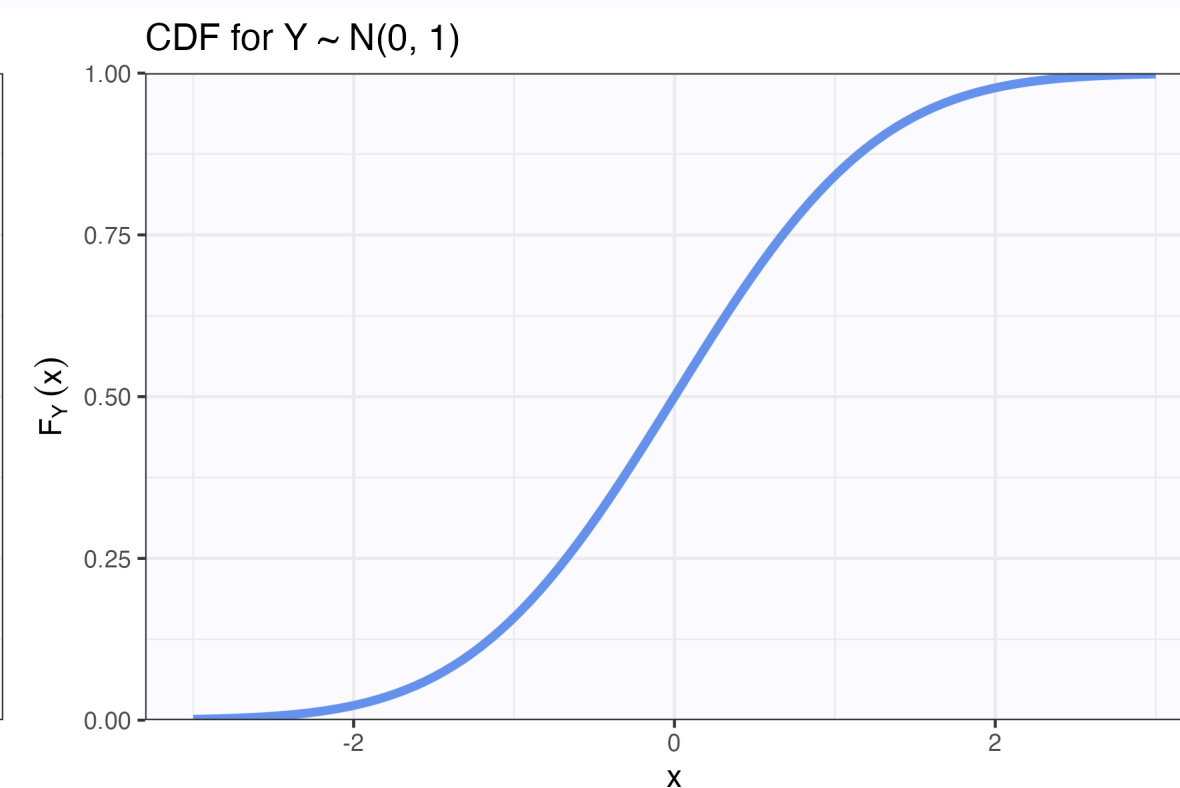
- The CDF is a step function and right-continuous.



Continuous RV

$$F_X(x) = \int_{-\infty}^x f_X(t) dt$$

- The CDF is continuous (left and right limits are equal).



Continuous CDF Example

Let $X \sim \text{Exp}(\lambda)$, with pdf

$$f_X(x) = \lambda e^{-\lambda x} I_{[0, \infty)}(x).$$

 **EXAMPLE**
a) Find $F_X(x)$. b) Use the CDF to calculate $\mathbb{P}(X \leq 10)$ when $\lambda = 1/5$.

$$a) F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x \lambda e^{-\lambda t} I_{[0, \infty)}(t) dt = \int_0^x \lambda e^{-\lambda t} dt = -e^{-\lambda t} \Big|_{t=0}^{t=x} = 1 - e^{-\lambda x}$$

$$F_X(x) = (1 - e^{-\lambda x}) I_{[0, \infty)}(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$b) F_X(10) = \mathbb{P}(X \leq 10) = (1 - e^{-(1/5)(10)}) I_{[0, \infty)}(10) = 0.864$$

Continuous CDF Example

Discrete CDF Example

EXAMPLE

Consider rolling one fair six-sided die, so that $\Omega = \{1, 2, 3, 4, 5, 6\}$, with $\mathbb{P}(\omega) = 1/6$ for each $\omega \in \Omega$. Let X be the number showing on the die. What is $F_X(x)$? Use this to find $\mathbb{P}(X \leq 5)$.

$$a) F_X(x) = \mathbb{P}(X \leq x) = \sum_{\omega \in \Omega, \omega \leq x} p(\omega) = \sum_{\omega \in \Omega, \omega \leq x} \frac{1}{6} = \frac{1}{6} |\{\omega \in \Omega, \omega \leq x\}|$$

↑ # elements

Where does CDF change? $x = 1, 2, 3, 4, 5, 6$

$$F_X(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 2/6 & 2 \leq x < 3 \\ \vdots & \vdots \\ 5/6 & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

↓

$$\begin{aligned} \rightarrow \mathbb{P}(X \leq 2.5) &= \mathbb{P}(X=1 \text{ or } X=2) \\ &= 1/6 + 1/6 \\ &= 2/6 \end{aligned}$$

$$b) \mathbb{P}(X \leq 5) = F_X(5) = 5/6$$

CDFs and Distributions

THEOREM

Let X be any random variable with CDF F_X . Let B be any subset of $\mathbb{R}$. Then $\mathbb{P}(X \in B)$ can be determined solely from F_X .

- The CDF contains all the information about the distribution of X .
- It doesn't matter whether X is discrete, continuous, or anything else.

CDFs and Distributions

COROLLARY

Let X be an absolutely continuous random variable with cumulative distribution function F_X . Let

$$\overset{\text{PDF}}{\underbrace{f_X(x)}} = \frac{d}{dx} \underbrace{F_X(x)}_{\text{CDF}} = F'(x).$$

Then, $f(x)$ is the density function for X .

Aside: The discrete case involves finding the differences between consecutive CDF values.

(Discrete case)

$$P(X=5) = P(X \leq 5) - P(X < 5)$$

$$= P(X \leq 5) - P(X \leq 4)$$

$$= F_X(5) - F_X(4)$$

$$= 5/6 - 4/6$$

$$= 1/6$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

CDFs and Distributions

EXERCISE: FROM CDF TO PDF

Suppose a continuous RV has CDF:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

- a. What is $\mathbb{P}(X \leq 0.75)$?
- b. Find the PDF $f_X(x)$.

a) $P(X \leq 0.75) = F_X(0.75) = (0.75)^2 = 0.5625$

b) $f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} = 2x I_{[0,1]}(x)$

CDFs and Distributions

CDFs and Distributions

EXERCISE: FROM CDF TO PMF

Suppose a discrete RV has CDF:

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 0.1 & 1 \leq x < 2 \\ 0.35 & 2 \leq x < 3 \\ 0.7 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

$$f_X(x) = \begin{cases} 0.1 & x=1 \\ 0.25 & x=2 \\ 0.35 & x=3 \\ 0.3 & x=4 \\ 0 & \text{otherwise} \end{cases}$$

a. What is $\mathbb{P}(X \leq 3.5)$? $= 0.7$

b. Find the PMF $f_X(x)$.

where probability changes: $x=1, 2, 3, 4$

$$\begin{aligned} P(X=3) &= P(X \leq 3) - P(X < 3) \\ &= F_X(3) - F_X(2) \\ &= 0.7 - 0.35 = 0.35 \end{aligned}$$

CDFs and Distributions

Mixture Distributions

PROPOSITION

Let $X_1, X_2, \dots$ be a random variables with CDFs $F_{X_1}, F_{X_2}, \dots$

For any constants p_i such that $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$, $F_G(x) = \sum_{i=1}^k p_i F_{X_i}(x)$ is the CDF of the mixture of F_{X_i} .

↳ p_i act as weights

$$F_G(x) = p_1 F_1(x) + p_2 F_2(x)$$

Mixture Distributions

💡 EXAMPLE

Suppose a bag contains two coins. You choose one at random and flip it once. Let X be the number of heads.

Coin A is a fair coin, with $\mathbb{P}(\text{flip Heads}) = 0.5$ Coin B is a loaded coin, with $\mathbb{P}(\text{flip Heads}) = 0.9$

Write out the CDF for the number of heads flipped.

Each coin has an equal prob. of being chosen: $P_A = 0.5$

$P_B = 0.5$

Possible values of heads: 0, 1

Case 0: $X < 0$ If a coin is flipped, we can't get < 0 heads.

$F_G(x) = 0$ when $x < 0$

Mixture Distributions

Case 1: $0 \leq x < 1$

$$F_A(x) = P(X=0) = 0.5$$

for $0 \leq x < 1$

$$F_B(x) = P(X=0) = 1 - 0.9 = 0.10$$

$$\begin{aligned}\therefore F_G(x) &= p_A F_A(x) + p_B F_B(x) \\ &= 0.5(0.5) + 0.5(0.10) \\ &= 0.3\end{aligned}$$

Case 2: $x \geq 1$

The number of heads can only be 0 or 1.
from $x \geq 1$, all possible outcomes are captured.

let $a \geq 1$

$$P(X \leq a) = ?$$

$$P(X \leq 1) = 1$$

$$P(X \leq 10) = 1$$

$$\therefore F_A(x) = 1 \quad \text{when } x \geq 1$$

$$F_B(x) = 1 \quad \text{when } x \geq 1$$

$$F_G(x) = 0.5(1) + 0.5(1) = 1$$

Mixture Distributions

$$F_G(x) = \begin{cases} 0 & x < 0 \\ 0.3 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Mixture Distributions

* Grace will post solutions (as practice problem)

Consider the scores of a midterm. Suppose that there are three types of students, modeled as follows:

1. Poorly prepared students who did not study much (30%): $X_1 \sim \mathcal{N}(\mu = 50, \sigma = 10)$.
2. Well prepared students who studied a lot (60%): $X_2 \sim \mathcal{N}(\mu = 80, \sigma = 8)$.
3. Students who didn't take the exam at all (10%): $X_3 = 0$ with probability 1.

EXERCISE: MIXTURE DISTRIBUTION FOR MIDTERM

Find the CDF of the overall score distribution X . Write your answer in terms of the standard normal CDF $\Phi(\cdot)$.

Hint: If $Y \sim \mathcal{N}(\mu, \sigma)$, then $F_Y(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$.

Mixture Distributions

Upcoming Assessments:

- Checkpoint 4 (Sections 2.5 - 2.6: Continuous RV and CDFs) is on Friday Oct 30th
- Checkpoint 5 (Sections 2.7 - 2.8: Joint, Marginal, and Conditional Distributions) is on Friday Nov 6th