

Lecture 12

Expected Values and Variance

Grace Tompkins

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Final Exam Information

Warning

Our final exam is scheduled for June 22nd, 2026 at 8:30am. Please find the room location on Workday.

The exam is 2.5 hours, with the **exact same rules as the midterm.**

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define and calculate expected values, variance, and standard deviation from discrete and continuous distributions

1 Expected Values

Expected Value of Random Variables

DEFINITION

The **expected value** of a random variable $g(X)$ is defined by

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x)p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x)f_X(x)dx & \text{if } X \text{ is absolutely continuous} \end{cases}$$

provided that the sum or integral exists.

- The sum in the discrete case is over all x such that $p_X(x) > 0$ (countable).
- The sum/integral exists when $\mathbb{E}[|g(X)|] < \infty$. Otherwise, we say that $\mathbb{E}[g(X)]$ does not exist.
- Note that you do **not** need to know the distribution/PMF/PDF/CDF of $g(X)$ to compute $\mathbb{E}[g(X)]$, only the distribution of X itself.

Expected Value of a Random Variable

💡 EXAMPLE

Let X be the value of the face of a die when rolled. What is the expected value of X ?

$$P(X=1) = P(X=2) = P(X=3) \dots P(X=6) = \frac{1}{6}$$

$$E(X) = \sum_{i=1}^6 x_i P_X(x_i)$$

$$= \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6)$$

$$= 3.5$$

→ this makes sense intuitively as 3.5 is halfway between 1 and 6 and we have equal probabilities for each event.

Expected Value of a Random Variable

💡 EXAMPLE

Let $X \sim \text{Binom}(n, \theta)$. What is $\mathbb{E}[X]$? Hint: we can use “kernel matching” and use the fact that

$$x \binom{n}{x} = n \binom{n-1}{x-1}.$$

$$\mathbb{E}(X) = \sum_{x=0}^n x \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= 0 + \sum_{x=1}^n x \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=1}^n n \binom{n-1}{x-1} \theta^x (1-\theta)^{n-x}$$

$$= n \sum_{y=0}^{n-1} \binom{n-1}{y} \theta^{y+1} (1-\theta)^{n-(y+1)}$$

since sum is 0 when $x=0$

using hint

substitute $x-1=y$
 $\Rightarrow x=y+1$

Expected Value of a Random Variable

$$= n \sum_{y=0}^{n-1} \binom{n-1}{y} \theta^{y+1} (1-\theta)^{n-(y+1)}$$

substitute $x-1=y$
 $\Rightarrow x=y+1$

$$= n \sum_{y=0}^{n-1} \binom{n-1}{y} \theta^y \theta (1-\theta)^{(n-1)-y}$$

$$= n\theta \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} \theta^y (1-\theta)^{(n-1)-y}}$$

PMF of Binom($n-1, \theta$)

$$= n\theta (1)$$

since \sum is over PMF

$$= n\theta$$

$$E(X) = n\theta \quad \text{when } X \sim \text{Binom}(n, \theta)$$

Expected Value of a Random Variable

EXERCISE: GAMMA EXPECTATION

Suppose $X \sim \text{Gamma}(\alpha, \lambda)$. What is $\mathbb{E}[X]$? Hint: kernel matching!

$$f_X(z) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\lambda z} I_{[0, \infty)}(z) \quad \alpha, \lambda > 0$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x) dx \\ &= \int_{-\infty}^{\infty} x \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{[0, \infty)}(x) dx \end{aligned}$$

Expected Value of a Random Variable

$$= \int_{-\infty}^{\infty} x \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{[0, \infty)}(x) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^\alpha e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \cdot \int_0^{\infty} \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x} dx$$

mult. by 1.

since PDF $\text{Gam}(\alpha+1, \lambda)$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha) \lambda} = \frac{\cancel{\alpha} \Gamma(\alpha)}{\cancel{\Gamma(\alpha)} \lambda} = \boxed{\frac{\alpha}{\lambda}}$$

since $\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$

Expected Value of a Random Variable

EXERCISE: MORE GAMMA EXPECTATIONS

Let $X \sim \text{Gam}(\alpha, \lambda)$ where $\alpha > 0$ and $\lambda > 0$. Recall that the PDF of a RV $Y \sim \text{Gam}(\alpha, \lambda)$ is given by

~~$f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} \mathbb{I}_{(0, \infty)}(y)$~~

$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathbb{I}_{(0, \infty)}(x)$

Let $t < \lambda$. Find $\mathbb{E}[\exp(tX)]$.

hint: kernel matching

Two options:

1) Find dist of $Z = e^{tX}$, then calculate $E(Z)$ (harder)

2) Apply def'n of $E(g(x)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
where $g(x) = e^{tx}$

Expected Value of a Random Variable

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx = \int_0^{\infty} e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{tx-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

looks like $\text{Gam}(\alpha, \lambda-t)$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} \int_0^{\infty} \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$$

mult by 1

$$= \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\lambda-t)^\alpha} (1)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^\alpha \text{ or } \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

Important Properties

(Where the expected value exists.)

Linearity

for any $a, b, c \in \mathbb{R}$, any functions g and h , and any random variables X and Y .

$$\mathbb{E}[ag(X) + bh(Y) + c] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(Y)] + c$$

Boundedness

If $a < g(x) < b$ for all x in the support of X , then $a < \mathbb{E}[g(X)] < b$.

Monotonicity

If $g(x) \leq h(x)$ for all x in the support of X , then $\mathbb{E}[g(X)] \leq \mathbb{E}[h(X)]$.

Independence

If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

only if X, Y are independent!!

Expected Value of A Function of Two Random Variables

EXERCISE: EXPECTATION OF TWO RV

Let $X \sim U(0, \theta)$ and $Y \sim \text{Exp}(1)$ be independent.

Find $\mathbb{E} \left[\frac{1}{2} (X + Y)^2 \right]$.

$$\begin{aligned} \mathbb{E} \left(\frac{1}{2} (X + Y)^2 \right) &= \mathbb{E} \left(\frac{1}{2} (X^2 + 2XY + Y^2) \right) \\ &= \frac{1}{2} \mathbb{E} (X^2 + 2XY + Y^2) \\ &= \frac{1}{2} \left[\mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) \right] \\ &= \frac{1}{2} \mathbb{E}(X^2) + \mathbb{E}(X)\mathbb{E}(Y) + \frac{1}{2} \mathbb{E}(Y^2) \end{aligned}$$

by linearity

by independence

Expected Value of A Function of Two Random Variables

$$E(X) = \int_0^{\theta} x \cdot \frac{1}{\theta} dx = \frac{1}{\theta} \int_0^{\theta} x dx = \frac{1}{\theta} \left. \frac{x^2}{2} \right|_0^{\theta} = \frac{\theta^2}{2\theta} = \frac{\theta}{2}$$

$$E(X^2) = \int_0^{\theta} x^2 \cdot \frac{1}{\theta} dx = \frac{1}{\theta} \frac{\theta^3}{3} = \frac{\theta^2}{3}$$

$$E(Y) = \int_0^{\infty} y \cdot \lambda e^{-\lambda y} dy = \lambda \int_0^{\infty} y e^{-\lambda y} dy = \dots = 1$$

$$E(Y^2) = \int_0^{\infty} y^2 \cdot \lambda e^{-\lambda y} dy = \Gamma(3) = 2$$

see if you can show this yourself!

$$E\left(\frac{1}{2}(X+Y)^2\right) = \frac{1}{2} \frac{\theta^2}{3} + \frac{\theta}{2} \cdot 1 + \frac{1}{2} \Gamma(3) = \frac{\theta^2}{6} + \frac{\theta}{2} + 1$$

↓_{2.1}

Scalar-valued Functions of Multiple Random Variables

THEOREM

Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function.

If X and Y are both discrete random variables, then

$$\mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) \underbrace{p_{X,Y}(x, y)}_{\text{joint PMF}}.$$

If X and Y are jointly absolutely continuous random variables, then

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \underbrace{f_{X,Y}(x, y)}_{\text{joint pdf}} dx dy.$$

Product of Expectations

THEOREM

- If X and Y are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$.

PROOF

Suppose that X and Y are jointly absolutely continuous random variables with joint PDF $f_{X,Y}(x, y)$. Note that $\mathbb{E}[g(X)h(Y)]$ is a scalar-valued function of X and Y .

$$\begin{aligned}\mathbb{E}[g(X)h(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) f(x,y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y) \cdot f(x) \cdot f(y) dx dy && \text{since } X, Y \text{ indep.} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)f(x) \cdot h(y)f(y) dx dy = \int_{-\infty}^{\infty} g(x)f(x) dx \int_{-\infty}^{\infty} h(y)f(y) dy \\ &= \mathbb{E}[g(X)]\mathbb{E}[h(Y)]\end{aligned}$$

Expectations from a Joint Distribution

EXERCISE: EXPECTATION OF A JOINT DISTRIBUTION

Let X and Y have joint PDF

$$f_{X,Y}(x, y) = 8xyI_{\{0 < x < y < 1\}}(x, y).$$

a. Calculate $\mathbb{E}[X]$.

↳ not nec independent

b. Calculate $\mathbb{E}[Y]$. (try at home)

c. Calculate $\mathbb{E}[XY]$.

$$\mathbb{E}(X) = \int_0^1 \int_0^y x \cdot 8xy \cdot dx dy = \int_0^1 \int_0^y 8x^2 y \, dx dy = \int_0^1 8y \frac{y^3}{3} dy = \frac{8}{3} \int_0^1 y^4 dy = \frac{8}{3} \cdot \frac{1}{5} = \frac{8}{15}$$

Expectations from a Joint Distribution

$$E(xy) = \int_0^1 \int_0^y xy \delta xy \cdot dx dy$$

$$= \int_0^1 \int_0^y \delta x^2 y^2 dx dy$$

$$= \int_0^1 \delta \left. \frac{x^3}{3} y^2 \right|_{x=0}^{x=y} dy$$

$$= \int_0^1 \frac{1}{3} y^3 y^2 dy = \int_0^1 \frac{1}{3} y^5 dy$$

$$= \frac{1}{3} \left. \frac{y^6}{6} \right|_{y=0}^{y=1} = \frac{1}{18} = \frac{1}{9}$$

Expectations from a Joint Distribution

2 Variance

Variance

DEFINITION

The **variance** of a random variable X is defined by

$$\begin{aligned}\sigma_X^2 &= \text{Var}(X) \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] && \text{(this is the "definition" ...)} \\ &= \underline{\mathbb{E}[X^2]} - \underline{(\mathbb{E}[X])^2} && \text{(... but this version is often easier for calculations)}\end{aligned}$$

- Careful here: $\mathbb{E}[X]$ is a number, not a random variable.
- The variance is a measure of the spread of the distribution of X around its mean $\mathbb{E}[X]$.
- Note that $g(X) = (X - \mathbb{E}[X])^2$ is a function of X , so we can compute $\text{Var}(X)$ using the definition of expected value.

Variance

- The “units” of $\text{Var}(X)$ are the square of the units of X , so sometimes we want to look at the spread of the distribution in the original units.

DEFINITION

The standard deviation of a random variable X is defined by

$$\sigma_X = \sqrt{\text{Var}(X)}.$$

asked to calc std deviation? Start by finding $\text{Var}(X)$

→ $E(X)$

→ $E(X^2)$

Properties of variance

(Where the variance exists.)

Scaling

For any $a \in \mathbb{R}$, $\text{Var}(aX) = a^2 \text{Var}(X)$.

Shift invariance

For any $a \in \mathbb{R}$, $\text{Var}(X + a) = \text{Var}(X)$.

Non-negativity

$\text{Var}(X) \geq 0$.

Relationship between $\text{Var}(X)$ and $\mathbb{E}(X^2)$

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2.\end{aligned}$$

$$\implies \text{Var}(X) \geq \mathbb{E}[X^2].$$

Variance

Let's do the following example together.

EXERCISE: UNIFORM VARIANCE

Let $X \sim \text{Unif}(0, 1)$. Find $\text{Var}(X)$.

Recall if $X \sim \text{Unif}(L, R)$, then

$$f_X(x; L, R) = \frac{1}{R - L} I_{[L, R]}(x)$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Variance

$$E(X) = \int_0^1 x \cdot \frac{1}{1-0} dx = \int_0^1 x dx = \left. \frac{x^2}{2} \right|_{x=0}^{x=1} = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 \cdot \frac{1}{1-0} dx = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_{x=0}^{x=1} = \frac{1}{3}$$

$$\begin{aligned} \boxed{\text{Var}(X)} &= E(X^2) - [E(X)]^2 \\ &= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \\ &= \frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \boxed{\frac{1}{12}} \end{aligned}$$

$$\begin{aligned} \sigma_X &= \sqrt{\text{VAR}(X)} \\ &= \sqrt{1/12} \end{aligned}$$

Variance

Variance

EXERCISE: EXPONENTIAL VARIANCE

Let $X \sim \text{Exp}(\lambda)$. Find $\text{Var}(X)$.

Hints: remember that $\mathbb{E}[X] = 1/\lambda$ and that $\Gamma(z) = (z-1)!$ for integer $z > 1$.

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{[0, \infty)}(x)$$

$$E(X) = 1/\lambda$$

$$E(X^2) = ?$$

$$\text{VAR}(X) = ?$$

hint: kernel matching ☺

Variance

$$E(X^2) = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = \int_0^{\infty} \lambda x^{3-1} e^{-\lambda x} dx$$

$$= \frac{\Gamma(3)}{\lambda^2} \int_0^{\infty} \frac{\lambda^2}{\Gamma(3)} \lambda x^{3-1} e^{-\lambda x} dx$$

mult by 1

$$= \frac{\Gamma(3)}{\lambda^2} \int_0^{\infty} \frac{\lambda^3}{\Gamma(3)} x^{3-1} e^{-\lambda x} dx$$


PDF of $\text{Gam}(3, \lambda)$

$$= \frac{\Gamma(3)}{\lambda^2} = \frac{2 \cdot 1}{\lambda^2} = \frac{2}{\lambda^2}$$

$$\boxed{\text{Var}(X)} = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda^2}\right) = \boxed{\frac{1}{\lambda^2}}$$

Variance

To Do

- Work on Assignment 3, due Wednesday June 10, 11:59pm on Gradescope.
- Read [Chapter 3.3 and 3.4](#)  before next class.