Lecture 12

Expected Values and Variance

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Final Exam Information

Warning

Our final exam is scheduled for June 22nd, 2026 at 8:30am. Please find the room location on Workday.

The exam is 2.5 hours, with the **exact same rules as the midterm**.

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define and calculate expected values, variance, and standard deviation from discrete and continuous distributions

1 Expected Values

Expected Value of Random Variables

📖 DEFINITION

The **expected value** of a random variable $g(X)$ is defined by

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x)p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x)f_X(x)\mathrm{d}x & \text{if } X \text{ is absolutely continuous} \end{cases}$$

provided that the sum or integral exists.

- The sum in the discrete case is over all x such that $p_X(x) > 0$ (countable).
- The sum/integral exists when $\mathbb{E}[|g(X)|] < \infty$. Otherwise, we say that $\mathbb{E}[g(X)]$ does not exist.
- Note that you do not need to know the distribution/PMF/PDF/CDF of $g(X)$ to compute $\mathbb{E}[g(X)]$, only the distribution of X itself.

Expected Value of a Random Variable

💡 **EXAMPLE**

Let X be the value of the face of a die when rolled. What is the expected value of X?

Expected Value of a Random Variable

💡 EXAMPLE

Let $X \sim \text{Binom}(n, \theta)$. What is $\mathbb{E}[X]$? Hint: we can use "kernel matching" and use the fact that $x\binom{n}{x} = n\binom{n-1}{x-1}$.

Expected Value of a Random Variable

Expected Value of a Random Variable

✏️ **EXERCISE: GAMMA EXPECTATION**

Suppose $X \sim \mathrm{Gamma}(\alpha, \lambda)$. What is $\mathbb{E}[X]$? Hint: kernel matching!

Expected Value of a Random Variable

Expected Value of a Random Variable

✏️ **EXERCISE: MORE GAMMA EXPECTATIONS**

Let $X \sim \mathrm{Gam}(\alpha, \lambda)$ where $\alpha > 0$ and $\lambda > 0$. Recall that the PDF of a RV $Y \sim \mathrm{Gam}(\theta, \beta)$ is given by

$$f_X(x) = \frac{\beta^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\beta x} I_{(0,\infty)}(x).$$

Let $t < \lambda$. Find $\mathbb{E}[\exp(tX)]$.

Expected Value of a Random Variable

Important Properties

(Where the expected value exists.)

Linearity

for any $a, b, c \in \mathbb{R}$, any functions g and h, and any random variables X and Y.

$$\mathbb{E}[ag(X) + bh(Y) + c] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(Y)] + c$$

Boundedness

If $a < g(x) < b$ for all x in the support of X, then $a < \mathbb{E}[g(X)] < b$.

Monotonicity

If $g(x) \leq h(x)$ for all x in the support of X, then $\mathbb{E}[g(X)] \leq \mathbb{E}[h(X)]$.

Independence

If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

Expected Value of A Function of Two Random Variables

✏️ **EXERCISE: EXPECATION OF TWO RV**

Let $X \sim U(0, \theta)$ and $Y \sim \text{Exp}(1)$ be independent.

Find $\mathbb{E}\left[\frac{1}{2}(X+Y)^2\right]$.

Expected Value of A Function of Two Random Variables

Scalar-valued Functions of Multiple Random Variables

THEOREM

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function.

If X and Y are both discrete random variables, then

$$\mathbb{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y).$$

If X and Y are jointly absolutely continuous random variables, then

$$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) \mathrm{d}x \mathrm{d}y.$$

Product of Expectations

📐 **THEOREM**

- If X and Y are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$.

📝 **PROOF**

Suppose that X and Y are jointly absolutely continuous random variables with joint PDF $f_{X,Y}(x, y)$. Note that $\mathbb{E}[g(X)h(Y)]$ is a scalar-valued function of X and Y.

Expectations from a Joint Distribution

✏️ **EXERCISE: EXPECTATION OF A JOINT DISTRIBUTION**

Let X and Y have joint PDF

$$f_{X,Y}(x, y) = 8xyI_{\{0<x<y<1\}}(x, y).$$

a. Calculate $\mathbb{E}[X]$.

b. Calculate $\mathbb{E}[Y]$.

c. Calculate $\mathbb{E}[XY]$.

Expectations from a Joint Distribution

Expectations from a Joint Distribution

2 Variance

Variance

📖 **DEFINITION**

The **variance** of a random variable X is defined by

$$\begin{aligned}\sigma_X^2 &= \mathrm{Var}(X) \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] && \text{(this is the "definition"...)} \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 && \text{(... but this version is often easier for calculations)}\end{aligned}$$

- Careful here: $\mathbb{E}[X]$ is a number, not a random variable.
- The variance is a measure of the spread of the distribution of X around its mean $\mathbb{E}[X]$.
- Note that $g(X) = (X - \mathbb{E}[X])^2$ is a function of X, so we can compute $\mathrm{Var}(X)$ using the definition of expected value.

Variance

- The "units" of $\mathrm{Var}(X)$ are the square of the units of X, so sometimes we want to look at the spread of the distribution in the original units.

📖 DEFINITION

The **standard deviation** of a random variable X is defined by

$$\sigma_X = \sqrt{\mathrm{Var}(X)}.$$

Properties of variance

(Where the variance exists.)

Scaling

For any $a \in \mathbb{R}$, $\operatorname{Var}(aX) = a^2 \operatorname{Var}(X)$.

Shift invariance

For any $a \in \mathbb{R}$, $\operatorname{Var}(X + a) = \operatorname{Var}(X)$.

Non-negativity

$\operatorname{Var}(X) \geq 0$.

Relationship between $\operatorname{Var}(X)$ and $\mathbb{E}(X^2)$

$$
\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2] \\
&= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\end{aligned}
$$

$$
\implies \operatorname{Var}(X) \geq \mathbb{E}[X^2].
$$

Variance

Let’s do the following example together.

> ✏️ **EXERCISE: UNIFORM VARIANCE**
>
> Let $X \sim \text{Unif}(0, 1)$. Find $\text{Var}(X)$.
>
> Recall if $X \sim Unif(L, R)$, then
>
> $$f_X(x; L, R) = \frac{1}{R - L} I_{[L,R]}(x)$$

Variance

Variance

Variance

✏️ **EXERCISE: EXPONENTIAL VARIANCE**

Let $X \sim \mathrm{Exp}(\lambda)$. Find $\mathrm{Var}(X)$.

Hints: remember that $\mathbb{E}[X] = 1/\lambda$ and that $\Gamma(z) = (z-1)!$ for integer $z > 1$.

Variance

Variance

To Do

- Work on Assignment 3, due Wednesday June 10, 11:59pm on Gradescope.
- Read Chapter 3.3 and 3.4 before next class.