Lecture 13

Change of Variables

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Use the Distribution method to transform discrete or continuous CDFs
- Use the Jacobian method to transform continuous PDFs

1 Transformations of CDFs

One-Dimensional Change of Variables

Sometimes we want to perform transformations on a random variable. This is when we can turn to **change of variables**.

- Let X be a random variable with some distribution
- Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$.
- We want to find the distribution of Y.

There are two ways to do this: the **distribution method** and the **Jacobian method**. We will present both.

One-Dimensional Change of Variables: Distribution Method

The **distribution method** just uses the definition:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}).$$

If we can characterize these sets, we can find the distribution of Y. This method always works, and is easy for discrete random variables.

THEOREM

Let X be a discrete random variable with PMF $p_X(x)$. Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$. Then the PMF of Y is given by

$$p_Y(y) = \sum_{x : g(x) = y} p_X(x) = \sum_{x \in g^{-1}(y)} p_X(x).$$

One-Dimensional Change of Variables: Distribution Method

Let's make this method a little more algorithmic.

If you want to find the distribution of $Y = g(X)$, then you can do the following:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}) = \cdots$$

1. Start with the definition of the distribution of Y.
2. Use the definition of Y in terms of X to rewrite the probability in terms of X.
3. Use the distribution of X to compute the probability that $g(X) \in A$.
4. Then play algebra games to get it into a nice form; possibly "match" a known PDF or CDF.

One-Dimensional Change of Variables: Distribution Method (Discrete)

Let's start with a straightforward example:

> 💡 **EXAMPLE**
>
> - Let $X \sim \text{Binom}(n, \theta)$, for some $\theta \in (0, 1)$.
> - Let $Y = n - X$. $\Rightarrow$ $X = n - Y$
>
> Find the PMF of Y.

$$f_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} I_{\{0,1,\ldots,n\}}(x)$$

(given on formula sheet)

One-Dimensional Change of Variables: Distribution Method (Discrete)

$X \sim Bin(n, \theta)$

$Y = n - X$

$\Rightarrow X = n - Y$

$$p_Y(y) = \sum_{x: g(x)=y} p_X(x) = \sum_{x: (n-x)=y} p_X(x) = p_X(n-y)$$

$$p_X(n-y) = \binom{n}{n-y} \theta^{n-y} (1-\theta)^{n-(n-y)} I_{\{0,1,\dots,n\}}(n-y)$$

NOTE: Symmetry of binom. coeff: $\binom{n}{k} = \binom{n}{n-k}$ $\binom{10}{9} = \binom{10}{1}$

$$\Rightarrow = \binom{n}{y} \theta^{n-y} (1-\theta)^{y} I_{\{0,1,\dots,n\}}(n-y)$$

One-Dimensional Change of Variables: Distribution Method (Discrete)

$$= \binom{n}{y} \theta^{n-y} (1-\theta)^{y} I_{\{0,1,\dots,n\}}(n-y)$$

$$= \binom{n}{y} \underbrace{(1-\theta)}_{\text{prob of success}}^{y} (1-(1-\theta))^{n-y} I_{\{0,1,\dots,n\}}(y) \qquad Y \sim \text{Bin}(n, (1-\theta))$$

Why did the indicator change?

$x \in \{0, 1, 2, \dots, n-1, n\}$

$y = n - x \Rightarrow y \in \{n, n-1, \dots, 2, 1, 0\}$

$\Rightarrow y \in \{0, 1, \dots, n\}$

One-Dimensional Change of Variables: Distribution Method (Continuous)

> 💡 **EXAMPLE**
>
> Let $X \sim \text{Exp}(\lambda)$, find the distribution of $Y = 3X$.

Start with the CDF!

CDF of X: $1 - e^{-\lambda x} I_{[0,\infty)}(x)$

$$P(Y \le y) = P(3X \le y)$$

$$= P(X \le y/3)$$

CDF of X →

$$= F_X(y/3)$$

$$= 1 - e^{-\lambda(y/3)} I_{[0,\infty)}(y)$$

$$= 1 - e^{-(\lambda/3)y} I_{[0,\infty)}(y) \Rightarrow \text{Exp}(\lambda/3)$$

$x = 0 \rightarrow y = 3x = 3(0) = 0$

$x = \infty \rightarrow y = \infty/3 = \infty$

↑ assessing bounds of $x = 0$ and $x = \infty$ at $g(x) = Y = 3X$

One-Dimensional Change of Variables: Distribution Method (Continuous)

> 💡 **EXAMPLE**
>
> If $X \sim \text{Exp}(\lambda)$, what is the CDF of $Y = X + 5$?

$$P(Y \le y) = P(X+5 \le y)$$
$$= P(X \le y-5)$$
$$= F_X(y-5)$$
$$= 1 - e^{-\lambda(y-5)} I_{[5,\infty)}$$

$X: [0, \infty)$

$y = x+5: [5, \infty)$

One-Dimensional Change of Variables: Distribution Method (Continuous)

PDF: $f_X(x) = 1 \, I_{[0,1]}(x)$

> ✏️ **EXERCISE: TRY THIS ONE**
>
> If $X \sim \text{Unif}(0, 1)$, what is the CDF of $Y = X + 3$? $(x = y - 3)$

① Start with CDF

when $x < 0$, $F_X(x) = 0$

when $0 \le x < 1$:

$$F_X(x) = \int_0^x 1 \, dt = x$$

when $x \ge 1$, $F_X(x) = 1$

when $x < 0 \Rightarrow y - 3 < 0$

$y < 3$ and $F_Y(y) = 0$

when $0 \le x < 1 \Rightarrow 0 \le y - 3 < 1$

$3 \le y < 4$

$F_Y(y) = F_X(y - 3) = (y - 3)$

when $x \ge 1 \Rightarrow y - 3 \ge 1 \Rightarrow y \ge 4$

$F_Y(y) = F_X(y - 3) = 1$

One-Dimensional Change of Variables: Distribution Method (Continuous)

✏️ **EXERCISE: TRY THIS ONE**

If $X \sim \text{Unif}(0, 1)$, what is the CDF of $Y = X + 3$?

Continued

$$F_Y(y) = \begin{cases} 0 & y < 3 \\ y - 3 & 3 \leq y < 4 \\ 1 & y \geq 4 \end{cases}$$

One-Dimensional Change of Variables: Useful Results

The following useful results follow from the previous examples

PROPOSITION

If X be a random variables with CDFs F_X Then the the following hold:

1. The RV $Y = X + c$ has CDF $F_X(x - c)$;
2. The RV $Y = kX$ has CDF $F_X(x/k)$ for any $k > 0$;

One-Dimensional Change of Variables: Jacobian Method

For **continuous** random variables, we can often use the **Jacobian method** in addition to the distribution method.

THEOREM

Let X be an (absolutely) continuous random variable, with density function f_X. Let $Y = h(X)$, where $h : \mathbb{R} \to \mathbb{R}$ is a function that is **differentiable and monotonic**.

Then Y is also absolutely continuous, and its density function f_Y is given by

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y}(h^{-1}(y)) \right|,$$

where $h^{-1}(y)$ is the unique number x such that $h(x) = y$.

One-Dimensional Change of Variables: Jacobian Method

EXAMPLE

- Let $X \sim \text{Unif}(0, 1)$.
- Let $Y = -\log(X)$.

Find the PDF of Y.

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$$

log (natural log) is monotonic + diff on $(0, \infty)$. ✓

① Find $h^{-1}(y)$

$h(x) = Y = -\log(x)$

$\Rightarrow x = e^{-Y}$

$\Rightarrow h^{-1}(y) = e^{-y}$

② Find $f_X(h^{-1}(y))$

2a: write pdf of x

$f_X(x) = 1 1_{(0,1]}(x)$

2b: $f_X(h^{-1}(y)) = 1 I_{(0,1]}(e^{-y})$

$e^{-y}: (0, 1]$

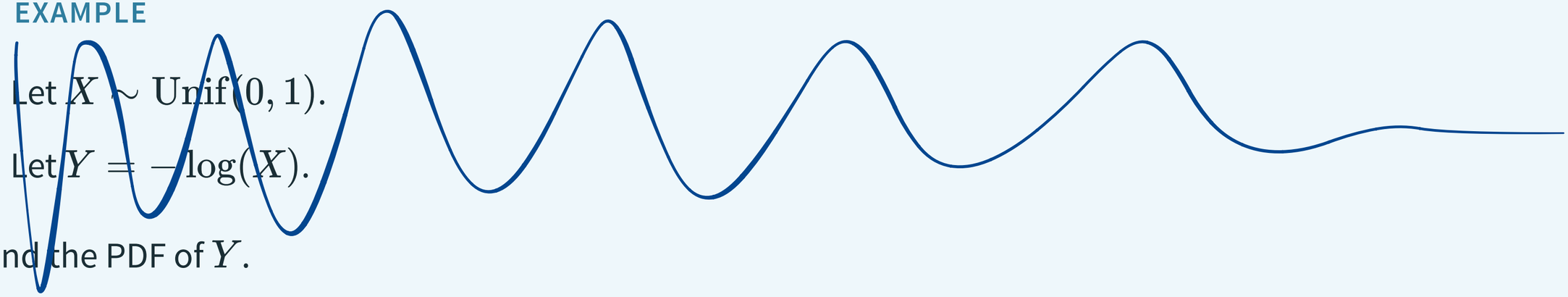
$y: (0, \infty)$

One-Dimensional Change of Variables: Jacobian Method

EXAMPLE

- Let $X \sim \text{Unif}(0, 1)$.
- Let $Y = -\log(X)$.

Find the PDF of Y.



Step 3 $\left|\frac{d}{dy} h^{-1}(y)\right| = \left|\frac{d}{dy} e^{-y}\right| = \left|-e^{-y}\right| = e^{-y}$

Step 4: Put it together!

$f_Y(y) = f_X(h^{-1}(y)) \left|\frac{d}{dy} h^{-1}(y)\right| = 1 \cdot e^{-y} I_{(0,\infty)} \Rightarrow \text{Exp}(\lambda = 1)$

One-Dimensional Change of Variables: Jacobian Method

✏️ **EXERCISE:**

1. Let $X \sim \text{Unif}(-1, 1)$. Use the distribution method to find the PDF of $Z = X^2$.
2. Let $X \sim \text{Gam}(\alpha, \lambda)$. Use the Jacobian method to find the PDF of $Y = 1/X$.

Recall the Gamma PDF:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

Recall Unif(a,b)

$$f_X(x) = \frac{1}{b-a} I_{[a,b]}(x)$$

One-Dimensional Change of Variables: Jacobian Method

1. $h(x)$ not monotonic on $(-1,1)$ ⇒ Dist'n method.

$h(x) = z = x^2$ $\quad x: [-1,1]$

$z: [0,1]$

For $z \in (0,1)$:

$$F_Z(z) = P(Z \le z) = P(X^2 \le z)$$

$$= P(-\sqrt{z} \le X \le \sqrt{z}) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{1-(-1)}\, dx = \frac{1}{2} x \Big|_{x=-\sqrt{z}}^{x=\sqrt{z}}$$

$$= \sqrt{z}\, I_{[0,1]}(z)$$ ← CDF

PDF: $f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \sqrt{z}\, I_{[0,1]}(z) = \frac{1}{2} z^{-1/2} I_{(0,1]}(z)$ ← can't ÷ z=0

One-Dimensional Change of Variables: Jacobian Method

2. $h(x) = 1/x$ is mon + diff :)

$h(x) = y = 1/x$

$x = 1/y$

$h^{-1}(y) = 1/y$

$$\left|\frac{d}{dy} h^{-1}(y)\right| = \left|\frac{d}{dy} y^{-1}\right|$$

$$= \left|\frac{-1}{y^2}\right|$$

$$= \frac{1}{y^2}$$

$$f_Y(y) = f_X(h^{-1}(y)) \left|\frac{d}{dy} h^{-1}(y)\right|$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\lambda(1/y)} \cdot \frac{1}{y^2} I_{(0,\infty)}(y)$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\lambda y} I_{(0,\infty)}(y)$$

$x: (0, \infty)$

$y: (0, \infty)$

↳ Inverse Gamma ☺

Upcoming Assessments:

- **Checkpoint 4 (Sections 2.4.1 - 2.5: Continuous Families and CDFs) is on Friday Nov 6th**
- **Checkpoint 5 (Sections 2.6 - 2.7: Change of Variables, Joint and Marginal Distributions) is on Friday Nov 6th**