Lecture 13

Change of Variables

Grace Tompkins

Last modified — 25 Oct 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Use the Distribution method to transform discrete or continuous CDFs
- Use the Jacobian method to transform continuous PDFs

1 Transformations of CDFs

One-Dimensional Change of Variables

Sometimes we want to perform transformations on a random variable. This is when we can turn to **change of variables**.

- Let X be a random variable with some distribution
- Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$.
- We want to find the distribution of Y.

There are two ways to do this: the **distribution method** and the **Jacobian method**. We will present both.

One-Dimensional Change of Variables: Distribution Method

The **distribution method** just uses the definition:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}).$$

If we can characterize these sets, we can find the distribution of Y. This method always works, and is easy for discrete random variables.



THEOREM

Let X be a discrete random variable with PMF $p_X(x)$. Let $Y = g(X)$ for some function $g : \mathbb{R} \to \mathbb{R}$. Then the PMF of Y is given by

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x) = \sum_{x \in g^{-1}(y)} p_X(x).$$

One-Dimensional Change of Variables: Distribution Method

Let's make this method a little more algorithmic.

If you want to find the distribution of $Y = g(X)$, then you can do the following:

$$\mathbb{P}(Y \in A) = \mathbb{P}(g(X) \in A) = \mathbb{P}(\{x : g(x) \in A\}) = \cdots$$

1. Start with the definition of the distribution of Y.
2. Use the definition of Y in terms of X to rewrite the probability in terms of X.
3. Use the distribution of X to compute the probability that $g(X) \in A$.
4. Then play algebra games to get it into a nice form; possibly "match" a known PDF or CDF.

One-Dimensional Change of Variables: Distribution Method (Discrete)

Let's start with a straightforward example:

> 💡 **EXAMPLE**
>
> - Let $X \sim \text{Binom}(n, \theta)$, for some $\theta \in (0, 1)$.
> - Let $Y = n - X$.
>
> Find the PMF of Y.

One-Dimensional Change of Variables: Distribution Method (Discrete)

One-Dimensional Change of Variables: Distribution Method (Continuous)

💡 EXAMPLE

Let $X \sim \text{Exp}(\lambda)$, find the distribution of $Y = 3X$.

One-Dimensional Change of Variables: Distribution Method (Continuous)

💡 EXAMPLE

If $X \sim \text{Exp}(\lambda)$, what is the CDF of $Y = X + 5$?

One-Dimensional Change of Variables: Distribution Method (Continuous)

✏️ **EXERCISE: TRY THIS ONE**

If $X \sim \text{Unif}(0, 1)$, what is the CDF of $Y = X + 3$?

One-Dimensional Change of Variables: Useful Results

The following useful results follow from the previous examples

PROPOSITION

If X be a random variables with CDFs F_X Then the the following hold:

1. The RV $Y = X + c$ has CDF $F_X(x - c)$;
2. The RV $Y = kX$ has CDF $F_X(x/k)$ for any $k > 0$;

One-Dimensional Change of Variables: Jacobian Method

For **continuous** random variables, we can often use the **Jacobian method** in addition to the distribution method.

THEOREM

Let X be an (absolutely) continuous random variable, with density function f_X. Let $Y = h(X)$, where $h : \mathbb{R} \to \mathbb{R}$ is a function that is **differentiable and monotonic**.

Then Y is also absolutely continuous, and its density function f_Y is given by

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{\mathsf{d}}{\mathsf{d}y}(h^{-1}(y)) \right|,$$

where $h^{-1}(y)$ is the unique number x such that $h(x) = y$.

One-Dimensional Change of Variables: Jacobian Method

💡 EXAMPLE

- Let $X \sim \text{Unif}(0, 1)$.
- Let $Y = -\log(X)$.

Find the PDF of Y.

One-Dimensional Change of Variables: Jacobian Method

✏️ **EXERCISE:**

1. Let $X \sim \text{Unif}\,(-1, 1)$. Use the distribution method to find the PDF of $Z = X^2$.
2. Let $X \sim \text{Gam}(\alpha, \lambda)$. Use the Jacobian method to find the PDF of $Y = 1/X$.

Recall the Gamma PDF:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

One-Dimensional Change of Variables: Jacobian Method

One-Dimensional Change of Variables: Jacobian Method

Upcoming Assessments:

- **Checkpoint 4 (Sections 2.4.1 - 2.5: Continuous Families and CDFs) is on Friday Nov 6th**
- **Checkpoint 5 (Sections 2.6 - 2.7: Change of Variables, Joint and Marginal Distributions) is on Friday Nov 6th**