

Lecture 13

Covariance, Correlation, and MGFs

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define covariance and correlation
- Calculate covariances and variances from discrete and continuous distributions
- Define and calculate a moment generating function
- Use the moment generating function to calculate moments

1 Covariance and Correlation

Covariance

If we have two random variables X and Y , we can measure the relationship between them.

DEFINITION

The **covariance** between two random variables X and Y is defined by

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

- $\text{Cov}(X, Y)$ is a scalar-valued function of X and Y .
- The covariance measures the **linear** relationship between X and Y .
- If $\text{Cov}(X, Y) > 0$, then X and Y tend to increase together.

Covariance

We can compute the covariance using the joint PMF/PDF of X and Y .:

$$\text{Cov}(X, Y) = \sum_x \sum_y (x - \mathbb{E}[X])(y - \mathbb{E}[Y])p_{X,Y}(x, y).$$

$$\text{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mathbb{E}[X])(y - \mathbb{E}[Y])f_{X,Y}(x, y) \, dx \, dy.$$

Properties of covariance

Linearity

For any $a, b, c \in \mathbb{R}$, and random variables X, Y , and Z ,

$$\text{Cov}(aX + bY, cZ) = a \text{Cov}(X, Z) + b \text{Cov}(Y, Z).$$

Easier calculation

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Independence

If X and Y are independent, then $\text{Cov}(X, Y) = 0$. The converse is false.

Covariance

EXERCISE: HEADS OR TAILS?

Let X be the number of heads in 5 tosses of a fair coin, and let Y be the number of tails in the same 5 tosses. Find $\text{Cov}(X, Y)$.

Hint: If $Z \sim \text{Binom}(n, \theta)$, then $\text{Var}(Z) = n\theta(1 - \theta)$.

Covariance

Variance, Covariance, and Sums

Let X and Y be random variables with finite variances.

- $$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y).$$

- If X and Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

- More generally, if X_1, \dots, X_n are independent random variables with finite variances, then

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i).$$

Correlation

Covariance is not a standardized measure of the relationship between X and Y .

For example, if we multiply X by 100, then $\text{Cov}(X, Y)$ will also be multiplied by 100.

DEFINITION

The **correlation** between two random variables X and Y is defined by

$$\rho_{XY} = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

- The correlation is a standardized measure of the linear relationship between X and Y .
- We'll see later that $-1 \leq \rho_{XY} \leq 1$.

Correlation

EXERCISE: TABULAR EXAMPLE

Let X and Y be discrete random variables with the following joint distribution:

$P(X, Y)$	$Y = 2$	$Y = 4$
$X = 1$	0.2	0.3
$X = 3$	0.1	0.4

Hint: $\sigma_X = 1$ and $\sigma_Y = 0.92$.

Calculate the correlation between X and Y .

Correlation

2 Moment generating functions

The Moment Generating Function

! Important

We will not cover/discuss probability generating functions or characteristic functions in this course. We will not discuss:

- (Beginning of 3.1) $r_X(t) = \mathbb{E}[t^X]$, the probability generating function of a random variable X .
- (Section 3.1.4) $c_X(t) = \mathbb{E}[e^{itX}]$, the characteristic function

The Moment Generating Function

DEFINITION

The **moment generating function** (MGF) of a random variable X is defined by

$$m_X(t) = \mathbb{E}[e^{tX}].$$

- The MGF is a scalar-valued function of t .

The Moment Generating Function

💡 EXAMPLE

Last lecture, we saw that $\mathbb{E}[\exp(tX)] = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$ when $X \sim \text{Gam}(\alpha, \lambda)$. (see “*Exercise: More Gamma Expectations*”). This was actually the moment generating function!

$$m_X(t) = \mathbb{E}[e^{tX}] = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}, \quad t < \lambda.$$

- $m_X(t)$ depends on the parameters α and λ of the distribution of X .
- It also depends on t , which is a free variable that we can choose.
- This seems like a weird object to care about, but it turns out to be very useful.

Using the MGF to Compute Moments

THEOREM

If X is a random variable with MGF $m_X(t)$, and there exists $s > 0$ such that, for all $t \in (-s, s)$, $m_X(t) < \infty$.

Then for any integer $k \geq 1$,

$$\mathbb{E}[X^k] = m_X^{(k)}(0) = \left. \frac{d^k}{dt^k} m_X(t) \right|_{t=0}.$$

- $\mathbb{E}[X^k]$ is called the k -th moment of X .
- The MGF is called the “moment generating function” because we can use it to compute the moments of X .
- Specifically, the first moment is $\mathbb{E}[X] = m'_X(0)$, and the second moment is $\mathbb{E}[X^2] = m''_X(0)$.

Using the MGF to Compute Moments

 EXAMPLE

Let $X \sim \text{Gam}(\alpha, \lambda)$ with MGF $m_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$. Find $\text{Var}(X)$.

Using the MGF to Compute Moments

Using the MGF to Compute Moments

 EXERCISE: MEAN AND VARIANCE OF NORMAL

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then,

$$m_X(t) = \mathbb{E}[e^{tX}] = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Find $\mathbb{E}[X]$ and $\text{Var}(X)$ using the MGF of X .

Using the MGF to Compute Moments

Sums of Independent Random Variables

THEOREM

Let X and Y be independent random variables with MGFs $m_X(t)$ and $m_Y(t)$, respectively.

Then the MGF of $X + Y$ is given by

$$\begin{aligned} m_{X+Y}(t) &= \mathbb{E}[e^{t(X+Y)}] = \mathbb{E}[e^{tX} e^{tY}] = \mathbb{E}[e^{tX}] \mathbb{E}[e^{tY}] && X \text{ and } Y \text{ are independent} \\ &= m_X(t) m_Y(t). \end{aligned}$$

Sums of Independent Random Variables

- We saw before how to find the PMF/PDF of $X + Y$ using convolution. The MGF gives us an alternative way to find the distribution of $X + Y$.
- If X_1, \dots, X_n are independent random variables with common MGF $m_X(t)$, then the MGF of $S_n = \sum_{i=1}^n X_i$ is given by

$$m_{S_n}(t) = (m_X(t))^n.$$

- If X has MGF $m_X(t)$, then $aX + b$ has MGF $e^{bt}m_X(at)$ for any $a, b \in \mathbb{R}$.

Uniqueness of the MGF

THEOREM

If X and Y are random variables with MGFs $m_X(t)$ and $m_Y(t)$, respectively, and there exists $s > 0$ such that for all $t \in (-s, s)$, $m_X(t) = m_Y(t) < \infty$, then X and Y have the same distribution.

This is a very important result, as it allows us to identify the distribution of a random variable by finding its MGF.

Using These Theorems Together

EXERCISE: SUMS OF NORMALS

Let X_1, \dots, X_n be independent identically distributed (i.i.d.) random variables with $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for all i .

Note that $m_{X_i}(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$ for $i = 1, \dots, n$.

Find the distribution of $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ using MGFs.

Using These Theorems Together

Functions of Non-Independent Random Variables

The previous theorems assume independence. Consider the scenario where X and Y are **not** independent.

COROLLARY

Let X and Y be random variables that are not necessarily independent.

Then the MGF of $h(X, Y)$ is given by

$$m_{h(X,Y)}(t) = \mathbb{E}[e^{t \times h(X,Y)}]$$

For example, if we were interested in the MGF of $X + Y$, we would solve $m_{X+Y}(t) = \mathbb{E}[e^{t(X+Y)}]$ directly.

Functions of Non-Independent Random Variables

EXERCISE: NONINDEPENDENT SUMS


Let X and Y have joint pmf:

$X \backslash Y$	0	1
0	0.1	0.4
1	0.4	0.1

- Find the MGF of $Z = X + Y$.
- Find $\mathbb{E}[Z]$ using the MGF.

Functions of Non-Independent Random Variables

To Do

- Work on Assignment 3, due Wednesday June 10, 11:59pm on Gradescope.
- Read [Chapter 3.5 and 3.6](#)  before next class.