Lecture 14

Joint Distributions

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate and utilize joint distributions

1 Joint Distributions

Joint Distributions

- Suppose X and Y are two random variables.
- We may be interested in their distributions separately.
- But this ignores the relationship between X and Y.

📖 **DEFINITION**

The joint CDF of the two random variables X and Y is defined by

$$F_{X,Y}(a,b) = \mathbb{P}(X \leq a, \ Y \leq b).$$

"and"

Recall:

$$\{X \leq a\,,\ Y \leq b\} = \{X \leq a\} \cap \{Y \leq b\}.$$

Joint PMFs and PDFs

Let X and Y be two random variables with joint CDF $F_{X,Y}(x, y)$.

📖 **DEFINITION**

If X and Y are both **discrete**, then the joint PMF of (X, Y) is defined by

$$p_{X,Y}(a, b) = \mathbb{P}(X = a,\ Y = b).$$

"and"

📖 **DEFINITION**

The random variables X and Y are jointly **(absolutely) continuous** if there exists a density function $f_{X,Y}(x, y)$ such that for any set $A \subset \mathbb{R}^2$, we have

$$\mathbb{P}\left((X, Y) \in A\right) = \iint_A f_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y.$$

↳ be careful of the bounds of the integral.

Joint PMF

> 💡 **EXAMPLE**
>
> - Consider the experiment of rolling two fair dice.
> - Let X be the lowest of the two rolls, Y be the highest.
>
> Find the joint PMF of X and Y.

This will involve finding all of the joint probabilities for each combination of the dice:

$X=3, Y=5$

$f_{X,Y}(x=3, y=5) = 2/36$

→ the probability that the lowest roll is 3 and highest is 5 (of two dice rolls)

$f_{X,Y}(x,y)$	Y= 1	2	3	4	5	6
Y= 1	1/36	2/36	2/36	2/36	2/36	2/36
X= 2	0	1/36	2/36	2/36	2/36	2/36
3	0	0	1/36	2/36	2/36	2/36
4	0	0	0	1/36	2/36	2/36
5	0	0	0	0	1/36	2/36
6	0	0	0	0	0	1/36

Joint PMF

✏️ **EXERCISE: FIND C**

Suppose $\mathbb{P}(X = x, Y = y) = c \cdot (x + 2y + 1) I_{\{0,1,2\}}(x) I_{\{0,1\}}(y)$

a. What value of c will make this a valid joint PMF?

b. What is $\mathbb{P}(X = 0, Y = 1)$?

(x,y)	$P(X=x, Y=y)$	
$(0,0)$	$c(0+2(0)+1) = c$	$1/18$
$(0,1)$	$c(0+2(1)+1) = 3c$	$3/18$
$(1,0)$	$2c$	$2/18$
$(1,1)$	$4c$	$4/18$
$(2,0)$	$3c$	$3/18$
$(2,1)$	$5c$	$5/18$

a) Total Prob: $\sum_x \sum_y P(x,y) = 1$

$18c = 1$

$\Rightarrow c = 1/18$

b) $P(X=0, Y=1) = 3/18 = 1/6$

Joint PMF

Joint PDF

💡 EXAMPLE

Let X and Y be jointly continuous, with joint density function

$$f(x,y) = \begin{cases} 4x^2y + 2y^5 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that this is a density function.

b. What is $\mathbb{P}(0 \leq X \leq 0.5, 0 \leq y \leq 0.75)$?

a) i) $f(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$ holds as $4x^2y + 2y^5 \geq 0$ for $0 \leq x \leq 1$, $0 \leq y \leq 1$ and 0 otherwise. ✓

ii) $\int_x \int_y f_{X,Y}(x,y)\,dy\,dx = 1$

(next page)

Joint PDF

$$\int_x \int_y (4x^2y + 2y^5) I_{[0,1]}(x) I_{[0,1]}(y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (4x^2y + 2y^5)\, dy\, dx = \int_0^1 \left(\frac{4x^2y^2}{2} + \frac{2y^6}{6} \right) \Big|_{y=0}^{y=1} dx$$

$$= \int_0^1 \left(2x^2 + \frac{1}{3}\right) dx = \left(\frac{2x^3}{3} + \frac{1}{3}x \right) \Big|_{x=0}^{x=1} = \frac{2}{3} + \frac{1}{3} = 1 \quad \checkmark$$

$\therefore$ This is a valid joint PDF.

b) $P(0 \le X \le 0.5, 0 \le Y \le 0.75) = \int_0^{0.5} \int_0^{0.75} (4x^2y + 2y^5)\, dy\, dx$

$$= \int_0^{0.5} \left(\frac{4x^2y^2}{2} + \frac{2y^6}{6} \right) \Big|_{y=0}^{y=0.75} dx$$

Joint PDF

b) $P(0 \le X \le 0.5, 0 \le Y \le 0.75) = \int_0^{0.5} \int_0^{0.75} (4x^2y + 2y^5)\,dy\,dx$

$= \int_0^{0.5} \left(\frac{4x^2y^2}{2} + \frac{2y^6}{6} \right) \Big|_{y=0}^{y=0.75} dx$

$= \int_0^{0.5} \left(\frac{9}{8}x^2 + \frac{243}{4096} \right) dx$

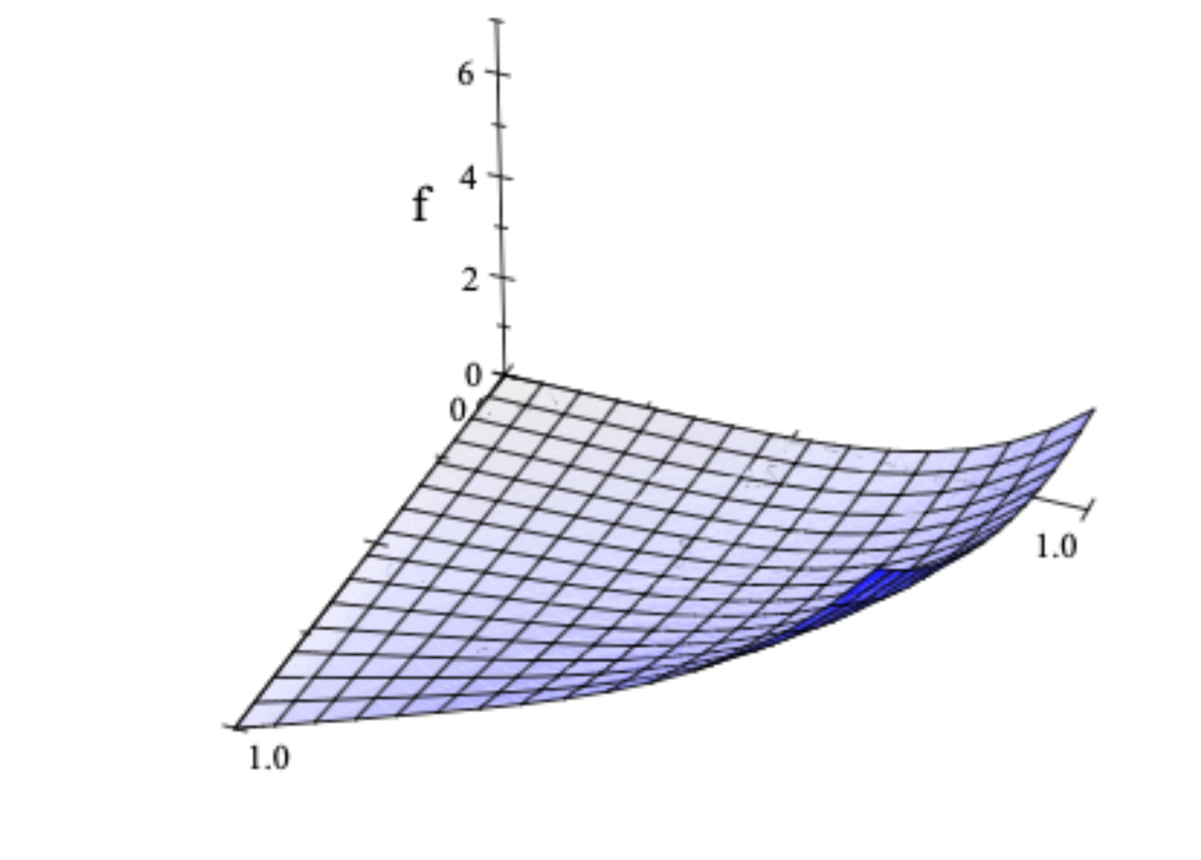
$= \left(\frac{9}{8}\frac{x^3}{3} + \frac{243}{4096}x \right) \Big|_{x=0}^{x=0.5}$

$= \boxed{0.0765}$

Joint PDF

The joint PDF is a surface:

$$f(x,y) = \begin{cases} 4x^2y + 2y^5 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Bivariate Normal distribution

📖 DEFINITION

Bivariate Normal Distribution: Given $\mu_1, \mu_2, \sigma_1, \sigma_2, \rho \in \mathbb{R}$ where $\sigma_1, \sigma_2 > 0$ and $-1 \leq \rho \leq 1$, the bivariate normal distribution $\mathcal{N}(\mu, \mu_2, \sigma_1, \sigma_2)$, is given by:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right\}.$$

When $\mu_1 = \mu_2 = \rho = 0$ and $\sigma_1 = \sigma_2 = 1$, this is the standard bivariate normal distribution.

Joint PDF

✏️ **EXERCISE: PROBABILITY OF X LESS THAN Y**

Let $f_{X,Y}(x,y) = e^{-x-y} I_{[0,\infty)}(x) I_{[0,\infty)}(y)$

What is the probability $\mathbb{P}(X < Y)$?

$$P(X<Y) = \int_0^\infty \int_0^y e^{-x} e^{-y} \, dx \, dy$$

$$= \int_0^\infty e^{-y} \left(\int_0^y e^{-x} \, dx \right) dy$$

$$= \int_0^\infty e^{-y} \left(-e^{-x} \Big|_{x=0}^{x=y} \right) dy$$

$$= \int_0^\infty e^{-y} \left(-e^{-y} - (-e^{-0}) \right) dy$$

$$= \int_0^\infty e^{-y} \left(1 - e^{-y} \right) dy$$

$$= \int_0^\infty \left(e^{-y} - e^{-2y} \right) dy$$

$$= -e^{-y} - \frac{e^{-2y}}{-2} \Big|_{y=0}^{y=\infty}$$

$$= 1 - 1/2 = 1/2$$

Joint PDF

Checkpoint 4

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing calculator
- Writing materials

Phones and laptops put away, please! You can have 15 minutes.

Upcoming Assessments:

- **Checkpoint 5 (Sections 2.7 - 2.8: Joint, Marginal, and Conditional Distributions) is on Friday Nov 6th**