Lecture 14

Conditional Expectations and Inequalities

Grace Tompkins

Last modified — 06 May 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate conditional expectations from conditional and joint distributions
- Use inequalities to find bounds of expectations and variances

1 Conditional Expectation

Conditional Expectation

📖 **DEFINITION**

If X and Y are two random variables, then the **conditional expectation** of X given $Y = y$ is

$$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\mathrm{d}x, \qquad \mathbb{E}[X|Y=y] = \sum_{x} x p_{X|Y}(x|y).$$

continuous

discrete

- This is the same definition we saw previously for expectation, just with the **conditional distribution**.

$$E(X^2|Y) = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx \quad \text{or} \quad \sum_{x} x^2 p_{X|Y}(x|y)$$

Conditional Expectation

💡 **EXAMPLE**

Let X and Y be random variables with joint density $f(x, y) = 2$ for $0 < x < 1, 0 < y < x$. What is the conditional expectation of Y given X?

Step 1: Find

$$f_{Y|X}(y, x) = \frac{f(x,y) \leftarrow \text{have}}{f_X(x) \leftarrow \text{need this}}$$

$$\boxed{f_X(x) = \int_0^x 2\,dy = 2x\, I_{(0,1)}(x)}$$

for $0 < x < 1$

$$f_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x} I_{(0,x)}(y) \underbrace{I_{(0,1)}(x)}_{\text{optional}}$$

Conditional Expectation

Step 2:

$$E(Y|X) = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy$$

$$= \int_{0}^{x} y \cdot \frac{1}{x} \, dy$$

$$= \frac{1}{x} \cdot \frac{1}{2} y^2 \Big|_{y=0}^{y=x}$$

$$= \frac{x^2}{2x}$$

$$= \frac{x}{2} \quad \text{for } 0 < x < 1$$

Aside:

$$E(Y|X=1/2) = \frac{(1/2)}{2} = \frac{1}{4}$$

Conditional Variance

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f_X(x)\, dx$$

📖 **DEFINITION**

If X and Y are two random variables, then the **conditional variance** of X given $Y = y$ is

$$\mathrm{Var}(X|Y=y) = \int_{-\infty}^{\infty} (x - \mathbb{E}[X|Y=y])^2 f_{X|Y}(x|y)\mathrm{d}x,$$

$$\mathrm{Var}(X|Y=y) = \sum_x (x - \mathbb{E}[X|Y=y])^2 p_{X|Y}(x|y).$$

$$Var(X|Y) = E(X^2|y) - (E(X|Y))^2$$

↳ this is often easier to calculate!

Conditional Variance

✏️ **EXERCISE: TRY AT HOME**

Let X and Y be random variables with joint density $f(x, y) = 2$ for $0 < x < 1, 0 < y < x$. What is the conditional variance of Y given X?

Try this on your own.

$$\text{Var}(Y|X=x) = E(Y^2|X=x) - (E(Y|X=x))^2$$

recall:

$$f_{Y|X}(y|x) = \frac{1}{x} I_{(0,x)}(y) \quad \text{for} \quad 0 < x < 1$$

$$E[Y|X] = \frac{x}{2}$$

$$E[Y^2|X] = \int_0^x y^2 \frac{1}{x}\,dy = \frac{y^3}{3} \cdot \frac{1}{x} \Big|_{y=0}^{y=x} = \frac{x^3}{3} \cdot \frac{1}{x} = \frac{x^2}{3}$$

$$\Longrightarrow$$

Conditional Variance

$$\mathrm{Var}(Y \mid X=x) = E(Y^2 \mid X=x) - \left(E(Y \mid X=x)\right)^2$$

$$= \left(\frac{x^2}{3}\right) - \left(\frac{x}{2}\right)^2$$

$$= \frac{x^2}{3} - \frac{x^2}{4}$$

$$= \frac{4x^2 - 3x^2}{12} = \frac{x^2}{12}$$

Conditional Expectation and Variance

Sometimes we are directly given information about the conditional distribution. If this is a “known” distribution, we can just use the properties of that distribution.

> 💡 **EXAMPLE**
>
> - Let $\Theta \sim \text{Unif}(0, 1)$
> - Let $Y|\Theta = \theta \sim \text{Binom}(n, \theta)$
>
> What are $\mathbb{E}[Y|\Theta = \theta]$ and $\text{Var}(Y|\Theta = \theta)$?

Because the conditional expecation follows $\text{Binom}(n, \theta)$, we know:

- $\mathbb{E}[Y|\Theta = \theta] = n\theta$ and
- $\text{Var}(Y|\Theta = \theta) = n\theta(1 - \theta)$.

This is much easier than finding the PMF/PDF of Y.

Conditional Expectation and Variance

Important

Quick knowledge check. Are conditional expectations and variances random variables?

Yes!

Conditional Expectation and Variance

- The properties of **expectation** and **variance** that we have seen before also hold for conditional expectation and variance.

But there are some additional properties as well because $\mathbb{E}[X|\Theta]$ and $\mathrm{Var}[X|\Theta]$ are themselves random variables, and have their own distributions.

Let $\Theta \sim \mathrm{Unif}(0,1), \text{ and } Y|\Theta = \theta \sim \mathrm{Binom}(n, \theta)$

- $W = \mathbb{E}[Y|\Theta] = n\Theta$ is a random variable that depends on Θ.
- But $\Theta \sim \mathrm{Unif}(0,1)$, so $W \sim \mathrm{Unif}(0,n)$!
- Using the Jacobian method, we can show that the PDF of $V = \mathrm{Var}(Y|\Theta) = n\Theta(1-\Theta)$ is given by

$$f_V(v) = \frac{2}{n\sqrt{1-4v/n}}, \quad 0 < v < n/4.$$

Hierarchical Models

We refer to this general setup as a **hierarchical model**.

1. We first draw Θ from some distribution.
2. Then we draw Y from a distribution that depends on Θ.
3. We can find the distribution of $Y|\Theta$ as well as those of its expectation.

$Z \sim \text{Exp}(\boxed{\lambda}) \rightarrow E(Z)\boxed{= 1/\lambda}$

$X|\Lambda \sim \text{Exp}(1/\Lambda) \rightarrow E(X|\Lambda) = 1/(1/\Lambda) = \Lambda$

$Y \sim \text{Gam}(\alpha, \lambda) \rightarrow E(Y) = \alpha/\lambda$

✏️ **EXERCISE: HIERARCHICAL MODEL**

- Let $\Lambda \sim \text{Gam}(1, 2)$.
- Let $\underline{X|\Lambda \sim \text{Exp}(1/\Lambda)}$.

Find the distribution of $\underline{W = \mathbb{E}[X|\Lambda]}$ and $\mathbb{E}[W]$.

$W = E[X|\Lambda] = \Lambda$ so $W \sim \text{Gam}(1,2)$ [the same dis'n as Λ]

$E(W) = E(\Lambda) = 1/2$

Hierarchical Models

Law of Total Expectation

Using the definition of the joint distribution of X and Λ, we can show that

$$\begin{aligned} f_{X,\Lambda}(x,\lambda) &= f_{X|\Lambda}(x|\lambda)f_\Lambda(\lambda) \\ &= \frac{1}{\lambda}e^{-x/\lambda}\cdot\frac{1}{\Gamma(1)}\lambda e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda) \\ &= e^{-x/\lambda}e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda). \end{aligned}$$

Using our definition of Expectation, we can find $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \int_0^\infty\int_0^\infty xe^{-x/\lambda}e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda)\mathbf{d}\lambda\mathbf{d}x = \cdots.$$

$E(X) = E\left[E(X|Y)\right]$

That is:

$$\mathbb{E}[X] = \mathbb{E}[W] = \mathbb{E}[\mathbb{E}[X|\Lambda]].$$

↑ marginal expectation

↑ conditional expectation

Law of Total Expectation and Variance (Tower Property)

THEOREM

Let X and Y be two random variables. Then,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$$

and

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]).$$

- The first equation shows what we just saw, but it is general and holds for any X and Y.
- The second equation is a bit more complicated, but it is also very useful.
- It shows that the variance of X can be decomposed into two parts: the expected value of the conditional variance of X given Y, and the variance of the conditional expectation of X given Y.

Law of Total Expectation and Variance (Tower Property)

💡 **EXAMPLE**

Let X and U be random variables such that $U \sim Unif(0,1)$, and $\mathbb{E}[X \mid U] = 3U^2$. Find $\mathbb{E}[X]$.

$$E(X) = E\left[E(X \mid U)\right]$$

$$= E\left[3U^2\right]$$

$$= 3E(U^2)$$

$$E(U) = \frac{0+1}{2} = \frac{1}{2}$$

$$Var(U) = \frac{(1-0)^2}{12} = \frac{1}{12}$$

recall: $Var(U) = E(U^2) - (E(U))^2$

$$\frac{1}{12} = E(U^2) - (1/2)^2$$

$$\Rightarrow E(U^2) = 1/3$$

(or do integration of uniform PDF)

Law of Total Expectation and Variance (Tower Property)

$$E(X) = E\left[E(X \mid U)\right]$$

$$= E\left[3U^2\right]$$

$$= 3E(U^2)$$

$$= 3(1/3)$$

$$= 1$$

2 Inequalities

Markov's Inequality

THEOREM

Let X be a random variable with $\mathbb{P}(X \geq 0) = 1$. Then, for any $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Note that this implies that for any random variable Y,

$$\mathbb{P}(|Y| \geq a) \leq \mathbb{E}[|Y|]/a.$$

Proof of Markov's Inequality

📝 **PROOF**

Let $Z = aI_{[a,\infty)}(X)$. We have that $Z \leq X$ almost surely, and hence $\mathbb{E}[Z] \leq \mathbb{E}[X]$ by monotonicity of expectation. But

$$\begin{aligned}\mathbb{E}[X] \geq \mathbb{E}[Z] \\ &= a\mathbb{P}(Z = a) + 0\mathbb{P}(Z = 0) \\ &= a\mathbb{P}(Z = a) \\ &= a\mathbb{P}(X \geq a).\end{aligned}$$

$$\Rightarrow \frac{E(X)}{a} \leq P(X \geq a$$

Chebyshev's[1] Inequality

THEOREM

Let X be a random variable with finite mean μ.

Then, for any $a > 0$,

$$\mathbb{P}(|X - \mu| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2}.$$

Chebyshev's[1] Inequality

📝 **PROOF**

We have

$$\begin{aligned}\mathbb{P}(|X-\mu| \geq a) &= \mathbb{P}((X-\mu)^2 \geq a^2) \\ &\leq \frac{\mathbb{E}[(X-\mu)^2]}{a^2} && \text{(by Markov's ineq.)} \\ &= \frac{\mathrm{Var}(X)}{a^2}.\end{aligned}$$

Comparing Markov and Chebyshev

Let X be a non-negative random variable with mean μ and variance μ.

We want to examine the bounds on $\mathbb{P}((X-\mu)/\mu \geq 1)$ given by Markov's and Chebyshev's inequalities.

Markov's inequality gives

$$\mathbb{P}((X-\mu)/\mu \geq 1) = \mathbb{P}(X \geq 2\mu) \leq \frac{\mathbb{E}[X]}{2\mu} = \frac{\mu}{2\mu} = \frac{1}{2}.$$

Chebyshev's inequality gives

$$\mathbb{P}((X-\mu)/\mu \geq 1) = \mathbb{P}(X-\mu \geq \mu) \leq \mathbb{P}(|X-\mu| \geq \mu) \leq \frac{\mathrm{Var}(X)}{\mu^2} = \frac{\mu}{\mu^2} = \frac{1}{\mu}.$$

So for any random variable with mean μ and variance μ, Chebyshev's inequality gives a tighter bound whenever $\mu > 2$.

Binomial Bounds

$X \sim Bin(n, \theta) \to E(X) = n\theta \quad Var(X) = n\theta(1-\theta)$

✏️ **EXERCISE: BINOMIAL BOUNDS**

Suppose you flip a fair coin 100 times. Use Markov's and Chebyshev's inequalities to approximate the probability of seeing 60 or more heads.

X = number of heads $\sim Bin(100, 0.5)$

Markov: $P(X \geq 60) \leq \frac{E(X)}{60} = \frac{100(0.5)}{60} = 5/6 = 0.8333$

Chebyshev: $P(X \geq 60) = P(X - 50 \geq 10) \leq P(|X-50| \geq 10)$

$$\leq \frac{Var(X)}{10^2} = \frac{100(0.5)(0.5)}{100}$$

$$= 1/4 = 0.25$$

Binomial Bounds

Far Away Stars

- Suppose that a radio telescope can measure the distance to a star.
- But due to atmospheric conditions, instrumental error, and movements of the earth, each measurement is a random variable with mean μ light years (the true distance) and variance 4 (square) light years.
- An astronomer plans to take n independent measurements of the distance and use their average $\overline{X}_n$ as an estimate for the true distance.

✏️ **EXERCISE: FAR AWAY STARS**

$P(|\overline{X}_n - \mu| < 1)$

How many measurements (n) should the astronomer make if they want the probability of a mismeasurement larger than 1 light year to be no more than 0.01?

Hint: recall that $\mathbb{E}[\overline{X}_n] = \mathbb{E}[X_1]$ and $\mathrm{Var}(\overline{X}_n) = \mathrm{Var}(X_1)/n$.

Try using Chebyshev's with "unknown" n

Far Away Stars

By Chebyshev's: $P(|\bar{X}_n - \mu| > 1) \leq \frac{Var(X_i)/n}{1^2}$

$$\Rightarrow P(|\bar{X}_n - \mu| > 1) \leq \frac{4}{n}$$

$$\Rightarrow \frac{4}{n} \leq 0.01$$

$$\Rightarrow \frac{4}{0.01} \leq n$$

$$\Rightarrow 400 \leq n$$

Thus, the astronomer should take atleast 400 measurements

Cauchy Schwarz Inequality

📐 THEOREM

Cauchy Schwarz for random variables

Let X and Y be two random variables with finite second moments. Then,

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}.$$

📌 COROLLARY

Let X and Y be two random variables with finite second moments. Then,

$$|\operatorname{Cov}(X, Y)| \leq \sqrt{\operatorname{Var}(X)\ \operatorname{Var}(Y)}.$$

Jensen's Inequality

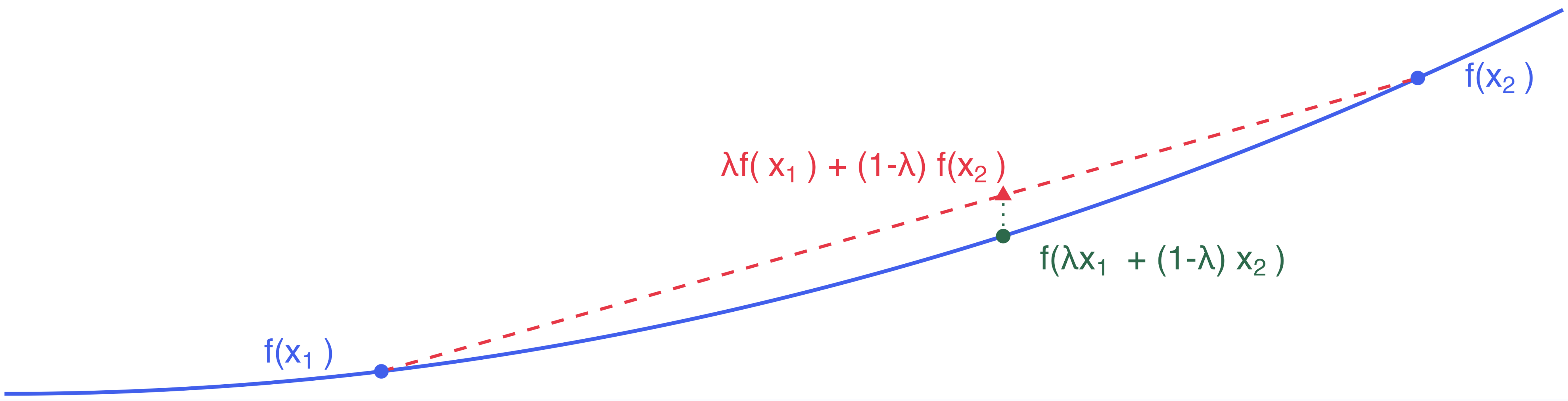
Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

THEOREM

Let X be a random variable with finite mean and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then,

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$



Questionable Friendships

✏️ **EXERCISE: DICE**

Your friend offers to play the following game with you.

1. Your friend pays you \$49 to roll 2 standard 6-sided dice.
2. If you see x pips, you pay your friend $\$x^2$.
3. Repeat as many times as you like, and your friend will keep paying you \$49 each time.

How many times should you play this game? Justify your answer.

hint! Let X be the sum of 2 dice

Questionable Friendships

Pay to Grace (friend)	Grace Pays You
49	-49
-121	$11^2 = 121$
49	-49
-16	16
49	-49
-81	$9^2 = 81$
49	-49
-100	$10^2 = 100$

122

Questionable Friendships

X = sum of two dice

$E(X) = 7$

$(E(X))^2 = 49$

$g(x) = x^2$ (convex on support)

$$\underbrace{E(x)^2}_{\text{roll dice}} = 49 \le \underbrace{E(X^2)}_{\text{payout.}}$$ by Jensen's

On average, you will lose money by playing this game. You <u>shouldn't</u> play! Expected payout < cost to play.

Followup on Jensen’s Inequality

✏️ **EXERCISE: VARIANCE OF JENSEN'S**

Show that the variance of a random variable is always non-negative.

$$Var(X) = E(X^2) - E(X)^2$$

By Jensen's, $f(x) = x^2$ is convex. Therefore,

$$E(x^2) \geq (E(x))^2$$

as $E(X^2) \geq (E(X))^2$, $\Rightarrow E(x^2) - (E(x))^2 \geq 0$

$$\Rightarrow Var(x) \geq 0$$

Followup on Jensen’s Inequality

To Do

- Work on Assignment 3, due TONIGHT June 10, 11:59pm on Gradescope.
- Read Chapter 4.2 - 4.3 before next class.