Lecture 14

Conditional Expectations and Inequalities

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate conditional expectations from conditional and joint distributions
- Use inequalities to find bounds of expectations and variances

1 Conditional Expectation

Conditional Expectation

📖 **DEFINITION**

If X and Y are two random variables, then the **conditional expectation** of X given $Y = y$ is

$$\mathbb{E}\left[X|Y=y\right] = \int_{-\infty}^{\infty} x f_{X|Y}\left(x|y\right)\mathrm{d}x, \qquad \mathbb{E}\left[X|Y=y\right] = \sum_{x} x p_{X|Y}(x|y).$$

- This is the same definition we saw previously for expectation, just with the **conditional distribution**.

Conditional Expectation

💡 EXAMPLE

Let X and Y be random variables with joint density $f(x, y) = 2$ for $0 < x < 1, 0 < y < x$. What is the conditional expectation of Y given X?

Conditional Expectation

Conditional Variance

📖 DEFINITION

If X and Y are two random variables, then the **conditional variance** of X given $Y = y$ is

$$\text{Var}(X|Y = y) = \int_{-\infty}^{\infty} (x - \mathbb{E}[X|Y = y])^2 f_{X|Y}(x|y) \mathrm{d}x,$$

$$\text{Var}(X|Y = y) = \sum_{x} (x - \mathbb{E}[X|Y = y])^2 p_{X|Y}(x|y).$$

Conditional Variance

✏️ **EXERCISE: TRY AT HOME**

Let X and Y be random variables with joint density $f(x, y) = 2$ for $0 < x < 1, 0 < y < x$. What is the conditional variance of Y given X?

Try this on your own.

Conditional Variance

Conditional Expectation and Variance

Sometimes we are directly given information about the conditional distribution. If this is a "known" distribution, we can just use the properties of that distribution.

> 💡 **EXAMPLE**
>
> - Let $\Theta \sim \text{Unif}(0, 1)$
> - Let $Y|\Theta = \theta \sim \text{Binom}(n, \theta)$
>
> What are $\mathbb{E}[Y|\Theta = \theta]$ and $\text{Var}(Y|\Theta = \theta)$?

Because the conditional expecation follows $\text{Binom}(n, \theta)$, we know:

- $\mathbb{E}[Y|\Theta = \theta] = n\theta$ and
- $\text{Var}(Y|\Theta = \theta) = n\theta(1 - \theta)$.

This is much easier than finding the PMF/PDF of Y.

Conditional Expectation and Variance

> **Important**
>
> Quick knowledge check. Are conditional expectations and variances random variables?

Conditional Expectation and Variance

- The properties of **expectation** and **variance** that we have seen before also hold for conditional expectation and variance.

But there are some additional properties as well because $\mathbb{E}[X|\Theta]$ and $\mathrm{Var}[X|\Theta]$ are themselves random variables, and have their own distributions.

Let $\Theta \sim \mathrm{Unif}(0,1), \text{ and } Y|\Theta = \theta \sim \mathrm{Binom}(n,\theta)$

- $W = \mathbb{E}[Y|\Theta] = n\Theta$ is a random variable that depends on Θ.
- But $\Theta \sim \mathrm{Unif}(0,1)$, so $W \sim \mathrm{Unif}(0,n)$!
- Using the Jacobian method, we can show that the PDF of $V = \mathrm{Var}(Y|\Theta) = n\Theta(1-\Theta)$ is given by

$$f_V(v) = \frac{2}{n\sqrt{1-4v/n}}, \quad 0 < v < n/4.$$

Hierarchical Models

We refer to this general setup as a **hierarchical model**.

1. We first draw Θ from some distribution.
2. Then we draw Y from a distribution that depends on Θ.
3. We can find the distribution of $Y|\Theta$ as well as those of its expectation.

✏️ **EXERCISE: HIERARCHICAL MODEL**

- Let $\Lambda \sim \mathrm{Gam}(1, 2)$.
- Let $X|\Lambda \sim \mathrm{Exp}(1/\Lambda)$.

Find the distribution of $W = \mathbb{E}[X|\Lambda]$ and $\mathbb{E}[W]$.

Hierarchical Models

Law of Total Expectation

Using the definition of the joint distribution of X and Λ, we can show that

$$\begin{aligned} f_{X,\Lambda}(x,\lambda) &= f_{X|\Lambda}(x|\lambda)f_{\Lambda}(\lambda) \\ &= \frac{1}{\lambda}e^{-x/\lambda}\cdot\frac{1}{\Gamma(1)}\lambda e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda) \\ &= e^{-x/\lambda}e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda). \end{aligned}$$

Using our definition of Expectation, we can find $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \int_0^\infty\int_0^\infty xe^{-x/\lambda}e^{-\lambda}I_{[0,\infty)}(x)I_{[0,\infty)}(\lambda)\mathrm{d}\lambda\mathrm{d}x = \cdots.$$

That is:

$$\mathbb{E}[X] = \mathbb{E}[W] = \mathbb{E}[\mathbb{E}[X|\Lambda]].$$

Law of Total Expectation and Variance (Tower Property)

THEOREM

Let X and Y be two random variables. Then,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$$

and

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]).$$

- The first equation shows what we just saw, but it is general and holds for any X and Y.
- The second equation is a bit more complicated, but it is also very useful.
- It shows that the variance of X can be decomposed into two parts: the expected value of the conditional variance of X given Y, and the variance of the conditional expectation of X given Y.

Law of Total Expectation and Variance (Tower Property)

💡 **EXAMPLE**

Let X and U be random variables such that $U \sim Unif(0.1)$, and $\mathbb{E}[X \mid U] = 3U^2$. Find $\mathbb{E}[X]$.

Law of Total Expectation and Variance (Tower Property)

2 Inequalities

Markov's Inequality

THEOREM

Let X be a random variable with $\mathbb{P}(X \geq 0) = 1$. Then, for any $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Note that this implies that for any random variable Y,

$$\mathbb{P}(|Y| \geq a) \leq \mathbb{E}[|Y|]/a.$$

Proof of Markov's Inequality

📝 PROOF

Let $Z = aI_{[a,\infty)}(X)$. We have that $Z \leq X$ almost surely, and hence $\mathbb{E}[Z] \leq \mathbb{E}[X]$ by monotonicity of expectation. But

$$\begin{aligned}\mathbb{E}[X] &\geq \mathbb{E}[Z] \\ &= a\mathbb{P}(Z = a) + 0\mathbb{P}(Z = 0) \\ &= a\mathbb{P}(Z = a) \\ &= a\mathbb{P}(X \geq a).\end{aligned}$$

Chebyshev's[1] Inequality

THEOREM

Let X be a random variable with finite mean μ.

Then, for any $a > 0$,

$$\mathbb{P}(|X - \mu| \geq a) \leq \frac{\text{Var}(X)}{a^2}.$$

Chebyshev's[1] Inequality

📝 **PROOF**

We have

$$\begin{aligned} \mathbb{P}(|X-\mu| \geq a) &= \mathbb{P}((X-\mu)^2 \geq a^2) \\ &\leq \frac{\mathbb{E}[(X-\mu)^2]}{a^2} && \text{(by Markov's ineq.)} \\ &= \frac{\text{Var}(X)}{a^2}. \end{aligned}$$

Comparing Markov and Chebyshev

Let X be a non-negative random variable with mean μ and variance μ.

We want to examine the bounds on $\mathbb{P}((X-\mu)/\mu \geq 1)$ given by Markov's and Chebyshev's inequalities.

Markov's inequality gives

$$\mathbb{P}((X-\mu)/\mu \geq 1) = \mathbb{P}(X \geq 2\mu) \leq \frac{\mathbb{E}[X]}{2\mu} = \frac{\mu}{2\mu} = \frac{1}{2}.$$

Chebyshev's inequality gives

$$\mathbb{P}((X-\mu)/\mu \geq 1) = \mathbb{P}(X-\mu \geq \mu) \leq \mathbb{P}(|X-\mu| \geq \mu) \leq \frac{\mathrm{Var}(X)}{\mu^2} = \frac{\mu}{\mu^2} = \frac{1}{\mu}.$$

So for any random variable with mean μ and variance μ, Chebyshev's inequality gives a tighter bound whenever $\mu > 2$.

Binomial Bounds

✏️ **EXERCISE: BINOMIAL BOUNDS**

Suppose you flip a fair coin 100 times. Use Markov's and Chebyshev's inequalities to approximate the probability of seeing 60 or more heads.

Binomial Bounds

Far Away Stars

- Suppose that a radio telescope can measure the distance to a star.
- But due to atmospheric conditions, instrumental error, and movements of the earth, each measurement is a random variable with mean μ light years (the true distance) and variance 4 (square) light years.
- An astronomer plans to take n independent measurements of the distance and use their average $\overline{X}_n$ as an estimate for the true distance.

> ✏️ **EXERCISE: FAR AWAY STARS**
>
> How many measurements should the astronomer make if they want the probability of a mismeasurement larger than 1 light year to be no more than 0.01?
>
> Hint: recall that $\mathbb{E}[\overline{X}_n] = \mathbb{E}[X_1]$ and $\mathrm{Var}(\overline{X}_n) = \mathrm{Var}(X_1)/n$.

Far Away Stars

Cauchy Schwarz Inequality

📐 THEOREM

Cauchy Schwarz for random variables

Let X and Y be two random variables with finite second moments. Then,

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}.$$

📌 COROLLARY

Let X and Y be two random variables with finite second moments. Then,

$$|\operatorname{Cov}(X,Y)| \leq \sqrt{\operatorname{Var}(X)\ \operatorname{Var}(Y)}.$$

Jensen's Inequality

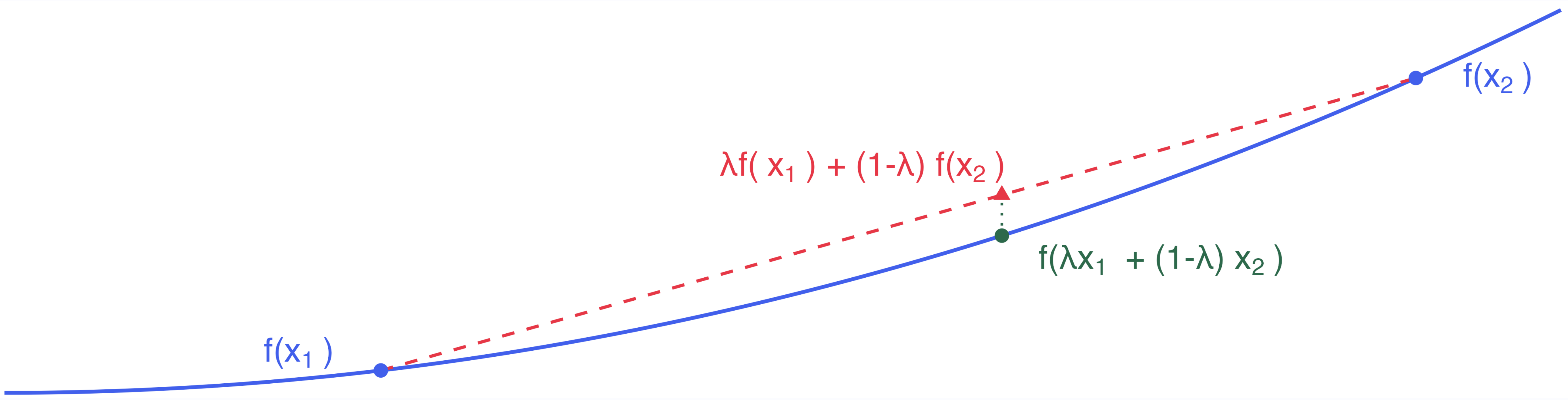
Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

THEOREM

Let X be a random variable with finite mean and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then,

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$



Questionable Friendships

✏️ **EXERCISE: DICE**

Your friend offers to play the following game with you.

1. Your friend pays you \$49 to roll 2 standard 6-sided dice.
2. If you see x pips, you pay your friend \$$x^2$.
3. Repeat as many times as you like, and your friend will keep paying you \$49 each time.

How many times should you play this game? Justify your answer.

Questionable Friendships

Followup on Jensen’s Inequality

✏️ **EXERCISE: VARIANCE OF JENSEN'S**

Show that the variance of a random variable is always non-negative.

Followup on Jensen's Inequality

To Do

- Work on Assignment 3, due TONIGHT June 10, 11:59pm on Gradescope.
- Read Chapter 4.2 - 4.3 before next class.