Lecture 15

Marginal Distributions and Independence

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- ~~Calculate and utilize joint distributions~~
- Determine whether random variables are independent based on their marginal and joint distributions
- Use the convolution method to calculate probabilities of the sum of independent RVs

1 Marginal Distributions

Marginal CDFs

Joint CDF: $F_{X,Y}(x,y) = P(X \le x, Y \le y)$
pmf: $p_{X,Y}(x,y)$
pdf: $f_{X,Y}(x,y)$

THEOREM

Let X and Y be two RV with joint CDF $F_{X,Y}$.

$$\lim_{a \to -\infty} F_{X,Y}(a,y) = 0 \qquad \forall y \in \mathbb{R}.$$
$$\lim_{b \to -\infty} F_{X,Y}(x,b) = 0 \qquad \forall x \in \mathbb{R}.$$
$$\lim_{a \to \infty,\, b \to \infty} F_{X,Y}(a,b) = 1.$$

DEFINITION

The **marginal CDFs** of X and Y are defined by

$$F_X(x) = \lim_{b \to \infty} F_{X,Y}(x,b), \qquad F_Y(y) = \lim_{a \to \infty} F_{X,Y}(a,y).$$

marg. CDF of X

ignore Y

Marginal PMFs and PDFs

THEOREM

If X and Y are **discrete**, then the marginal PMFs of X and Y are given by

$$p_X(x) = \sum_y p_{X,Y}(x,y), \qquad \text{and} \quad p_Y(y) = \sum_x p_{X,Y}(x,y).$$

THEOREM

If X and Y are **continuous**, then the marginal PDFs of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}y, \qquad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x.$$

Important

All of this generalizes to more than two random variables.

In this course, we will focus only on cases involving two random variables, since anything beyond that involves linear algebra (which is not a pre-req for this course)

Discrete Marginal Distribution

- Consider the experiment of rolling two fair dice. Let X be the lowest of the two rolls, Y be the highest.

$f_{X,Y}(x,y)$	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$	$Y=6$	$P(X=x)$
$X=1$	1/36	2/36	2/36	2/36	2/36	2/36	11/36
$X=2$	0	1/36	2/36	2/36	2/36	2/36	9/36
$X=3$	0	0	1/36	2/36	2/36	2/36	7/36
$X=4$	0	0	0	1/36	2/36	2/36	5/36
$X=5$	0	0	0	0	1/36	2/36	3/36
$X=6$	0	0	0	0	0	1/36	1/36
	1/36	3/36	5/36	7/36	9/36	11/36	

> 💡 **EXAMPLE**
>
> Use the joint PMF of X and Y to find the marginal PMFs of X and Y.

$P(X=1) = P(X=1 \cap Y=1) + P(X=1 \cap Y=2) \dots P(X=1 \cap Y=6)$

Discrete Marginal Distribution

$$P(X=x) = \begin{cases} 11/36 & x=1 \\ 9/36 & x=2 \\ \vdots & \\ 1/36 & x=6 \\ 0 & \text{otherwise} \end{cases}$$

$$p(Y=y) = \begin{cases} 1/36 & y=1 \\ \vdots & \\ 0 & \text{otherwise} \end{cases}$$

Sanity check!

$$\sum_x p(x) = 1$$

$$\sum_y p(y) = 1$$

Discrete Marginal Distribution

✏️ **EXERCISE: COFFEE SHOP**

A coffee shop records the orders of its customers. Let X denote the number of espresso shots ordered and Y denote the number of food items ordered, where $X \in \{0, 1, 2\}$ and $Y \in \{0, 1, 2\}$. The joint PMF is given by the following table:

$P(X = x, Y = y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.10	0.08	0.02
$X = 1$	0.15	0.20	0.10
$X = 2$	0.05	0.15	0.15

b)

$P(X = x)$	x
0.20	0
0.45	1
0.35	2
0	o/w

- Find $P(Y > 1, X \leq 1)$ (see last class) $= P(Y=2, X=0) + P(Y=2, X=1) = 0.02 + 0.10 = 0.12$
- Find the marginal PMF of X
- Calculate $\mathbb{P}(X \geq 1)$ $= P(X=1) + P(X=2)$
  $= 0.45 + 0.35$
  $= 0.80$

and ↓

Challenge: Find $F_X(x)$.

Discrete Marginal Distribution

Continuous Marginal Distributions

💡 EXAMPLE

Let X and Y be jointly continuous, with joint density function

$$f(x,y) = \begin{cases} 4x^2y + 2y^5 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $f_Y(y)$ and $f_X(x)$.

a) $f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 (4x^2y + 2y^5)\,dx = \frac{4x^3y}{3} + 2y^5x \Big|_{x=0}^{x=1}$

$= \frac{4}{3}y + 2y^5$ for $0 \le y \le 1$

$= \left(\frac{4}{3}y + 2y^5\right) I_{[0,1]}(y)$

Continuous Marginal Distributions

$$f_X(x) = \int_0^1 (4x^2y + 2y^5) I_{[0,1]}(x)\, dy$$

$$= \left(\frac{4x^2y^2}{2} + \frac{2y^6}{6} \right) I_{[0,1]}(x) \Big|_{y=0}^{y=1}$$

$$= \left(2x^2 + \frac{1}{3} \right) I_{[0,1]}(x)$$

Continuous Marginal Distributions

✏️ **EXERCISE: JOINT UNIFORM DISTRIBUTION**

Let X and Y be continuous random variables with PDF

$$f_{X,Y}(x,y) = I_{[0,1]}(x) I_{[0,1]}(y) = I_{[0,1]^2}(x,y).$$

a. Find $F(x,y) = \int_0^x \int_0^y f_{X,Y}(s,t)\mathrm{d}t\mathrm{d}s$ for $0 \leq x, y \leq 1$; → $0 \leq x \leq 1$, $0 \leq y \leq 1$

b. ~~Compute $\mathbb{P}(X \leq 0.3, Y \leq 0.8)$~~ Find $f_X(x)$

c. Calculate $\mathbb{P}(X - 2Y > 0)$.

c) Try on your own! Answer is 1/4

Continuous Marginal Distributions

$f_{X,Y}(s,t)$

a) $P(X \le x, Y \le y) = \int_0^x \int_0^y (1)\, dt\, ds$

$= \int_0^x t \Big|_{t=0}^{t=y} ds$

$= \int_0^x y\, ds$

$= y s \Big|_{s=0}^{s=x} = xy$

what this means:

For $0 \le x \le 1$
$0 \le y \le 1$

$P(X \le x, Y \le y) = xy$

b) Find $f_X(x) = \int I_{[0,1]}(x)\, I_{[0,1]}(y)\, dy$

$= \int_0^1 (1)\, dy\, I_{[0,1]}(x) = y \Big|_{y=0}^{y=1} I_{[0,1]}(x) = I_{[0,1]}(x)$

2 Independence

Independent Random Variables

THEOREM

X and Y are independent if and only if for any sets A and B we have

$$\mathbb{P}(X \in A,\ Y \in B) = \mathbb{P}(X \in A)\ \mathbb{P}(Y \in B).$$

$P(X \cap Y) = P(X)P(Y)$

This is the definition of independence. It says that the joint distribution factors into the product of the marginals.

But this has immediate consequences for the joint CDF, PMF, and PDF.

Choosing $A = (-\infty, x]$ and $B = (-\infty, y]$, we have that, if X and Y are independent, then

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y) = F_X(x)F_Y(y).$$

The converse is also true: if $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for all x, y, then X and Y are independent.

Independent Random Variables, Using PMFs

THEOREM

If X and Y are discrete random variables, then X and Y are independent if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y),$$

for all $x, y \in \mathbb{R}$.

Independent Random Variables, Using PDFs

THEOREM

If X and Y are jointly continuous random variables, then X and Y are independent if and only if their joint density can be chosen such that

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

for all $x, y \in \mathbb{R}$.

- Regardless of continuous or discrete, when you hear “independent”, think “joint factors into the product of the marginals”.

Independence

✏️ **EXERCISE: EVALUATING INDEPENDENCE USING MARGINAL DISTRIBUTIONS**

Let X and Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 8xy & \text{if } 0 \leq x < y < 1 \\ 0 & \text{else.} \end{cases}$$

Find the marginal PDFs of X and Y. Are X and Y independent?

$$f_X(x) = \int_x^1 8xy\,dy = 8x\frac{y^2}{2}\Big|_{y=x}^{y=1} = 4x(1-x^2)I_{[0,1)}(x)$$

$$f_Y(y) = \int_0^y 8xy\,dx = 8\frac{x^2}{2}y\Big|_{x=0}^{x=y} = 4y^3 I_{[0,1)}(y)$$

As $f(x)f(y) = 4x(1-x^2)4y^3 I_{[0,1)}(x)I_{[0,1)}(y) \neq f(x,y) = 8xy\cdots$ there are NOT independent.

Maximum/Minimum of Independent Random Variables

💡 EXAMPLE

Let X and Y be independent random variables with CDFs F_X and F_Y.

Let $W = \max\{X, Y\}$.

Find the CDF of W.

$$F_W(w) = P(W \le w) = P(\max\{X, Y\} \le w)$$
$$= P(X \le w, Y \le w)$$
$$= F_{X,Y}(w, w)$$
$$= F_X(w) F_Y(w)$$
$$(= P(X \le w) P(Y \le w))$$

Maximum/Minimum of Independent Random Variables

Sums of Independent Random Variables

To find the distribution of a sum of independent random variables, we could use the distribution method or the Jacobian method.

For this specific case where RVs are independent, there's also a third method called convolution.

THEOREM

$X+Y=u \Rightarrow X=u-Y$

$x=u-w$

Let X and Y be independent random variables.

If X and Y are both discrete random variables, then the PMF of $U=X+Y$ is given by

$x=u-w \quad y=w$

$$p_U(u)=\sum_w p_X(u-w)p_Y(w)=\sum_w p_Y(u-w)p_X(w).$$

If X and Y are both continuous random variables, then the PDF of $U=X+Y$ is given by

$$f_U(u)=\int_{-\infty}^{\infty} f_X(u-w)f_Y(w)\mathrm{d}w=\int_{-\infty}^{\infty} f_Y(u-w)f_X(w)\mathrm{d}w.$$

w's associated with Y.

Sums of Independent Random Variables

$\text{Unif}(a,b) = \frac{1}{b-a} I_{[a,b]}(x)$

1 0

> 💡 **EXAMPLE**
>
> Let X and Y be independent $\text{Unif}(0,1)$ random variables. Let $S = X + Y$. Find the distribution of S.

X; Unif(0,1) → $X:[0,1]$

Y; Unif(0,1) → $Y:[0,1]$

$S = X + Y$

$Y = S - X$

$S = X + Y$: $S:[0,2]$

$$f_S(s) = \int_{-\infty}^{\infty} f_X(x) f_Y(s-x)\, dx$$

$$= \int_{-\infty}^{\infty} I_{[0,1]}(x) I_{[0,1]}(s-x)\, dx$$

only non-zero where

$0 \le x \le 1$ AND $0 \le s-x \le 1$

Sums of Independent Random Variables

$0 \le x \le 1$ AND $0 \le s-x \le 1$ (intersection)

$0 \le x \le 1$ AND $s-1 \le x \le s$

This also is

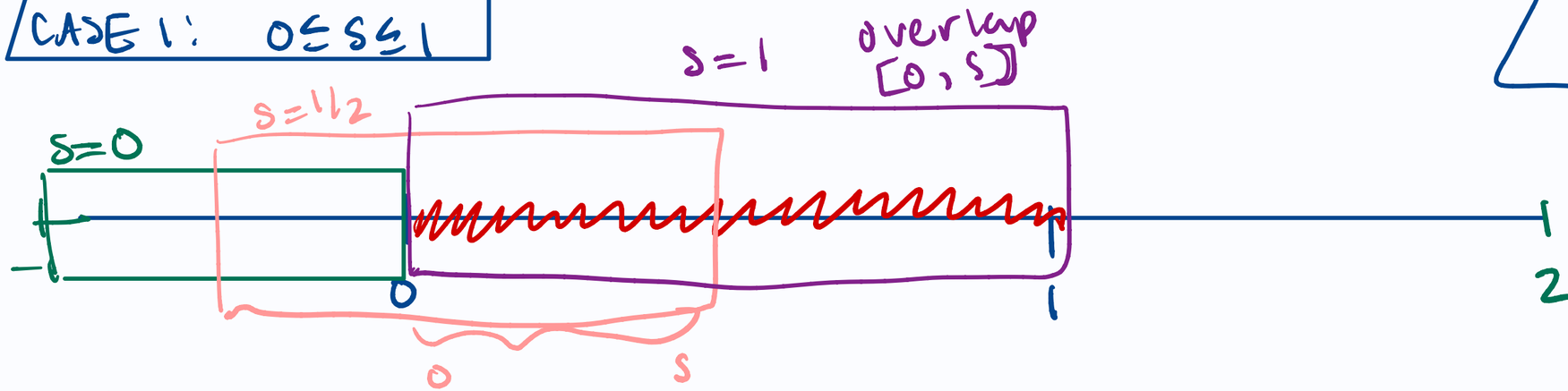
$[\max(0, s-1), \min(1, s)]$

$= [0, s]$ for $0 \le s \le 1$

or

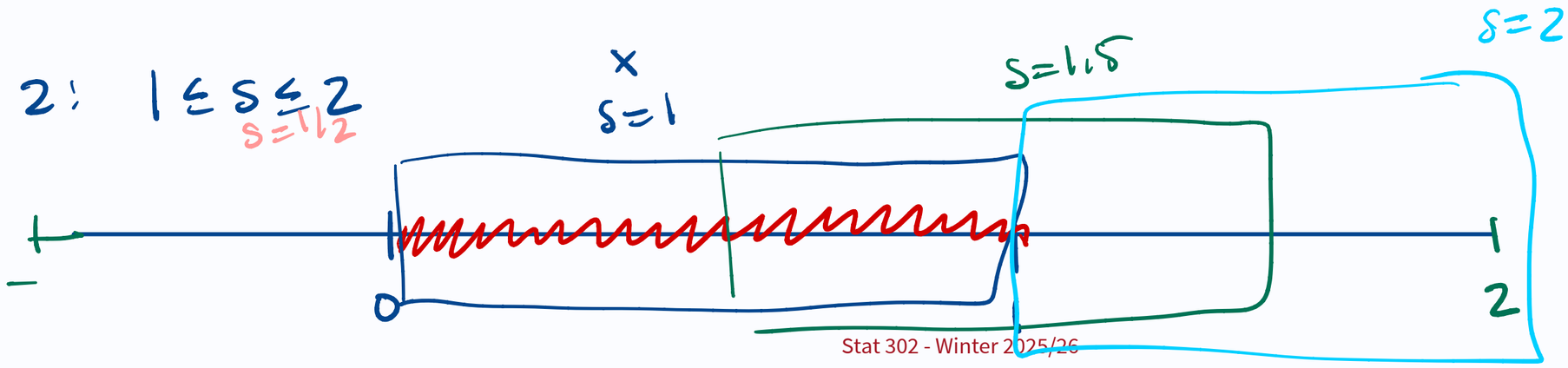
$[s-1, 1]$ for $1 \le s \le 2$

CASE 1: $0 \le s \le 1$



intersection of $0 \le x \le 1$ and $s-1 \le x \le s$ is $[0, s]$

CASE 2: $1 \le s \le 2$



overlap is $[s-1, 1]$

Sums of Independent Random Variables

$$f_S(s) = \begin{cases} \int_0^s 1\,dx = s & 0 \le s \le 1 \\ \int_{s-1}^1 1\,dx = 2-s & 1 \le s \le 2 \\ 0 & \text{o/w} \end{cases}$$

Upcoming Assessments:

2.6 – 2.7, Change of vars + joint distn)

- Checkpoint 5 (~~Sections 2.7 - 2.8: Joint, Marginal, and Conditional Distributions~~) is on Friday Nov 6th
- Midterm 2 (Sections 2.4 - 3.3) is on Friday November 20th