

# Lecture 15

Convergence, Part I

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Last modified — 06 May 2026

# Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define convergence in probability
- Determine when a sequence converges in probability
- Define and apply the Weak Law of Large Numbers

$\rightarrow$  <sup>+</sup>Convergence "almost surely"

# 1 Convergence in Probability

# Convergence (Calculus)

- In calculus, you looked at the limit of a sequence of numbers,  $a_n$ , as  $n$  goes to infinity.

## EXAMPLE

- $\lim_{n \rightarrow \infty} 1/n = 0$ ;
- $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$ ;
- $\lim_{n \rightarrow \infty} (1 + a/n)^n = e^a$  for all  $a \in \mathbb{R}$ ;
- $\lim_{n \rightarrow \infty} (1 + a_n/n)^n = e^a$  if  $a_n \rightarrow a$ .

- In probability, we look at the limit of a sequence of random variables,  $X_n$ , as  $n$  goes to infinity.
- This turns out to be more complicated, because there are different **modes of convergence**.

We will discuss 3 types of convergence.

# Convergence in Probability

## 📖 DEFINITION

A sequence of random variables  $X_1, X_2, \dots, X_n, \dots$  converges in probability to a random variable  $X$  if for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| < \epsilon) = 1.$$

- We can think of  $a_n = \mathbb{P}(|X_n - X| < \epsilon)$  as a sequence of numbers that goes to one as  $n$  goes to infinity.
- This is the most similar to limits of sequences of numbers.
- Can also be written as  $\mathbb{P}(|X_n - X| \geq \epsilon) \rightarrow 0$ . (complement)
- Common notation:  $X_n \xrightarrow{p} X$ .

↳ "X<sub>n</sub> converges in probability to X"

# Convergence in Probability

$$n = 1, 2, \dots$$

## EXAMPLE

Suppose  $\mathbb{P}(X_n = 1 - 1/n) = 1$  and  $\mathbb{P}(Y = 1) = 1$ . Show that the sequence  $\{X_n\}$  converges in probability to  $Y$ .

let  $\epsilon > 0$

$$|X_n - Y| = \left| 1 - \frac{1}{n} - 1 \right| = \left| -\frac{1}{n} \right| = \frac{1}{n}$$

$$P(|X_n - Y| < \epsilon) = P\left(\frac{1}{n} < \epsilon\right)$$

No matter how small  $\epsilon$  is, for any  $\epsilon > 0$ , we can always find a "n" such that  $\frac{1}{n} < \epsilon \Rightarrow \frac{1}{\epsilon} < n$

Ex: choose  $N > \frac{1}{\epsilon}$ ,  $\forall n > N \Rightarrow \frac{1}{n} < \epsilon$  will hold

# Convergence in Probability

Therefore,  $P(|X_n - Y| < \epsilon)$

$$= P(1/n < \epsilon) = 1$$

since

$\forall \epsilon > 0, \exists n$  where this holds.

Thus,

$$\lim_{n \rightarrow \infty} P(|X_n - Y| < \epsilon) = 1$$

$$\forall \epsilon > 0$$

$$\therefore X_n \xrightarrow{p} Y$$

# Convergence in Probability

## 💡 EXAMPLE

Let  $U \sim \text{Unif}(0, 1)$  and define

$$X_n = U + B_n, \quad \Rightarrow \quad B_n = X_n - U$$

where  $B_n \sim \text{Bern}(1/n)$  are independent Bernoulli random variables, also independent of  $U$ .

Show  $X_n \xrightarrow{p} U$ .

$$\begin{aligned} & P(|X_n - U| \geq \epsilon) \\ &= P(|B_n| \geq \epsilon) \end{aligned}$$

$$\epsilon > 0$$

↳ when is  $|B_n| > \epsilon$  where  $\epsilon > 0$ ?  $B_n = \{0, 1\}$

$$\leq P(B_n = 1) = \frac{1}{n} \quad \text{since } B_n \sim \text{Bern}(1/n)$$

$\epsilon = 1.5$   
 $P(|B_n| \geq 1.5) = 0$

# Convergence in Probability

$$\lim_{n \rightarrow \infty} P(|X_n - u| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0$$

$$\lim_{n \rightarrow \infty} P(|X_n - u| \geq \epsilon) = 0 \quad \therefore X_n \xrightarrow{P} u$$

# Convergence in Probability

## EXERCISE: MAXIMUM OF IID UNIFORMS

Let  $U_1, U_2, \dots$  be i.i.d. Unif(0, 1) random variables. Define  $Y_n = \max\{U_1, \dots, U_n\}$ .

Show that  $Y_n \xrightarrow{p} 1$ .

Hint:  $|Y_n - 1| > \epsilon$  if and only if  $Y_n < 1 - \epsilon$ .

$$\begin{aligned} P(|Y_n - 1| \geq \epsilon) &= P(|Y_n - 1| > \epsilon) \quad \text{since continuous} \\ &= P(\underline{Y_n < 1 - \epsilon}) \\ &= P(U_1 < 1 - \epsilon, U_2 < 1 - \epsilon, \dots, U_n < 1 - \epsilon) \\ &= \prod_{i=1}^n P(U_i < 1 - \epsilon) \\ &= (1 - \epsilon)^n \quad \text{because } P(U_i < 1 - \epsilon) = \int_0^{1-\epsilon} \underline{(1-x)} \, dx = (1 - \epsilon) \end{aligned}$$

# Convergence in Probability

$$\begin{aligned}\lim_{n \rightarrow \infty} P(|Y_n - 1| > \epsilon) &= \lim_{n \rightarrow \infty} (1 - \epsilon)^n \\ &= 0\end{aligned}$$

as  $(1 - \epsilon) \in (0, 1)$

$$\therefore Y_n \xrightarrow{p} 1$$

# Weak Law of Large Numbers (WLLN)

## THEOREM

Let  $X_1, X_2, \dots, X_n \dots$  be independent and identically distributed (i.i.d) random variables with finite mean  $\mu$ . Then,

$$\mu = E(X)$$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{p} \mu.$$

## Interpretation

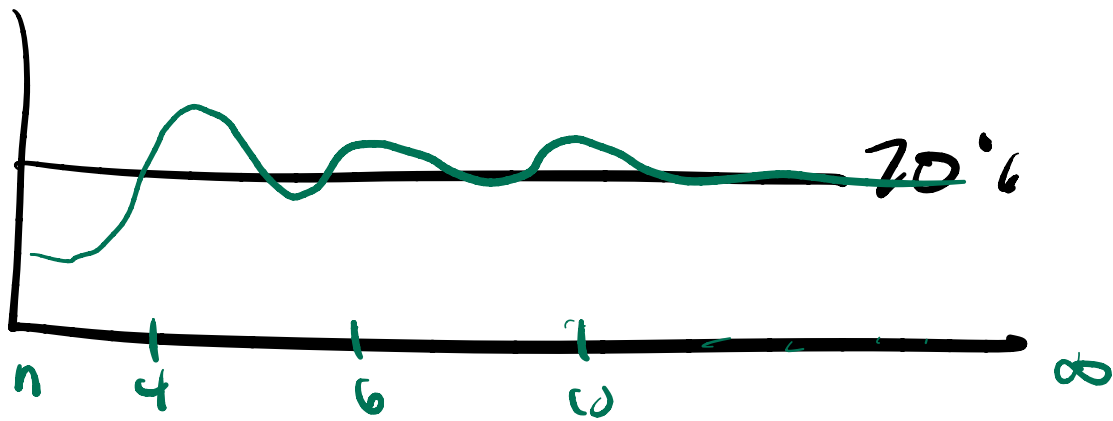
The distribution of  $\bar{X}_n$  gets more and more concentrated around  $\mu$  as  $n$  increases.

What % of the earth is covered by water?

Land Water  
 || ~~||||~~ ~~||||~~ ~~||||~~ |||

$n$	$\bar{x}$
4	0.5
6	0.667
11	0.54
$\vdots$	
20	0.65

$\mu$   
70%



# Weak Law of Large Numbers (WLLN)

## EXERCISE: DICE

Let  $X_n$  be the sum of the squares of  $n$  independent rolls of a fair six-sided die.

That is

$$X_n = \sum_{i=1}^n X_{n,i}^2,$$

where  $X_{n,i}$  is the result of the  $i$ -th die roll.

Show that  $n^{-1} X_n$   $\xrightarrow{p} m$  for some  $m$  (find  $m$  explicitly).

$$Y = \frac{1}{n} X_n = \frac{1}{n} \sum_{i=1}^n X_{n,i}^2 \rightarrow \text{this is an average of IID RVs.}$$

## Weak Law of Large Numbers (WLLN)

$$E(Y_{nii}) = E(X_{nii}^2) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) \\ = \frac{91}{6}$$

by WLLN

$$\frac{1}{n} \sum_{i=1}^n X_{i/n}^2 \xrightarrow{p} E(X_{nii}^2) = \frac{91}{6}$$

$$m = \frac{91}{6}$$

Aside: by WLLN,

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow E(X_i) = 3.5$$

# Proof of WLLN (with an extra condition)

## PROOF

- Assume that  $\text{Var}(X_i) = \sigma^2 < \infty$  (same for all  $i$ ). This is not required for the WLLN, but it makes the proof easier.

Then, by Chebyshev's inequality, for all  $\epsilon > 0$ ,

$$\mathbb{P} \left( \left| \bar{X}_n - \mu \right| \geq \epsilon \right) \leq \frac{\text{Var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2/n}{\epsilon^2} \rightarrow 0.$$

## 2 Convergence Almost Surely (with Probability One)

converge A.S.  $\Downarrow$   
converge w.p. 1

# Convergence Almost Surely (with Probability One)

## DEFINITION

A sequence of random variables  $X_1, X_2, \dots, X_n, \dots$  converges **almost surely (or w.p. 1)** to a random variable  $X$  if for all  $\epsilon > 0$ ,

$$\mathbb{P} \left( \lim_{n \rightarrow \infty} |X_n - X| < \epsilon \right) = 1.$$

- This is a stronger notion of convergence than convergence in probability.
- This is equivalent to saying that  $\mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1$ .
- Writing this statement more explicitly, we are really demanding that

$$\mathbb{P} \left( \left\{ \omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right\} \right) = 1.$$

- Common notation:  $X_n \xrightarrow{a.s.} X$ .

# Convergence Almost Surely (with Probability One)

## 💡 EXAMPLE

Let  $U \sim \text{Unif}(0, 1)$  and

$$X_n = \begin{cases} 3 & U \leq \frac{2}{3} - \frac{1}{n} \\ 8 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} 3 & U \leq \frac{2}{3} \\ 8 & \text{otherwise} \end{cases}$$

Does  $X_n \xrightarrow{a.s.} Y$ ?

$$u = 2/3$$

# Convergence Almost Surely (with Probability One)

Consider 3 cases

①  $u > 2/3$ ,  $X_n = 8$ ,  $Y_n = 8$

②  $u < 2/3$ ,  $Y = 3$ , and for large enough  $n$ ,  $u \leq 2/3 - 1/n$  will also hold. Thus,  $X_n = 3$ , too.

③  $u = 2/3$ ,  $Y = 3$ .

$X = 3$  when  $u \leq 2/3 - 1/n$

$2/3 \leq 2/3 - 1/n$

$0 \leq -1/n$  which is impossible for  $n \geq 1$

$\therefore X = 8$



## Convergence Almost Surely (with Probability One)

But because  $U \sim \text{Unif}(0,1)$  which is continuous,  
 $P(U = 2/3) = 0$ . So this case does not matter!

$$\therefore X_n \xrightarrow{\text{a.s.}} Y$$

# Convergence Almost Surely (with Probability One)

## 💡 EXAMPLE

Let  $Y \sim \text{Unif}(0, 1)$  and  $X_n = Y^n$ . Prove that  $X_n \xrightarrow{a.s.} 0$ .

$$\hookrightarrow 0 \leq y \leq 1$$

$$\textcircled{1} \quad 0 \leq y < 1 : \quad \lim_{n \rightarrow \infty} y^n = 0$$

$$\textcircled{2} \quad y = 1 : \quad \lim_{n \rightarrow \infty} y^n = \lim_{n \rightarrow \infty} (1)^n = 1$$

$\hookrightarrow P(Y=1) = 0$ , so this case doesn't matter.

$$X_n \xrightarrow{a.s.} 0$$

# Convergence Almost Surely (with Probability One)

# Convergence Almost Surely (with Probability One)

## 💡 EXAMPLE

Let  $U \sim \text{Uniform}(0, 1)$ . Define the sequence  $X_1, X_2, X_3, \dots$  by partitioning  $[0, 1]$  into successive blocks:

$$B_1 = [0, 1]; \quad B_2 = [0, \frac{1}{2}]; \quad B_3 = [\frac{1}{2}, 1]; \quad B_4 = [0, \frac{1}{3}]; \quad B_5 = [\frac{1}{3}, \frac{2}{3}]; \quad \dots$$

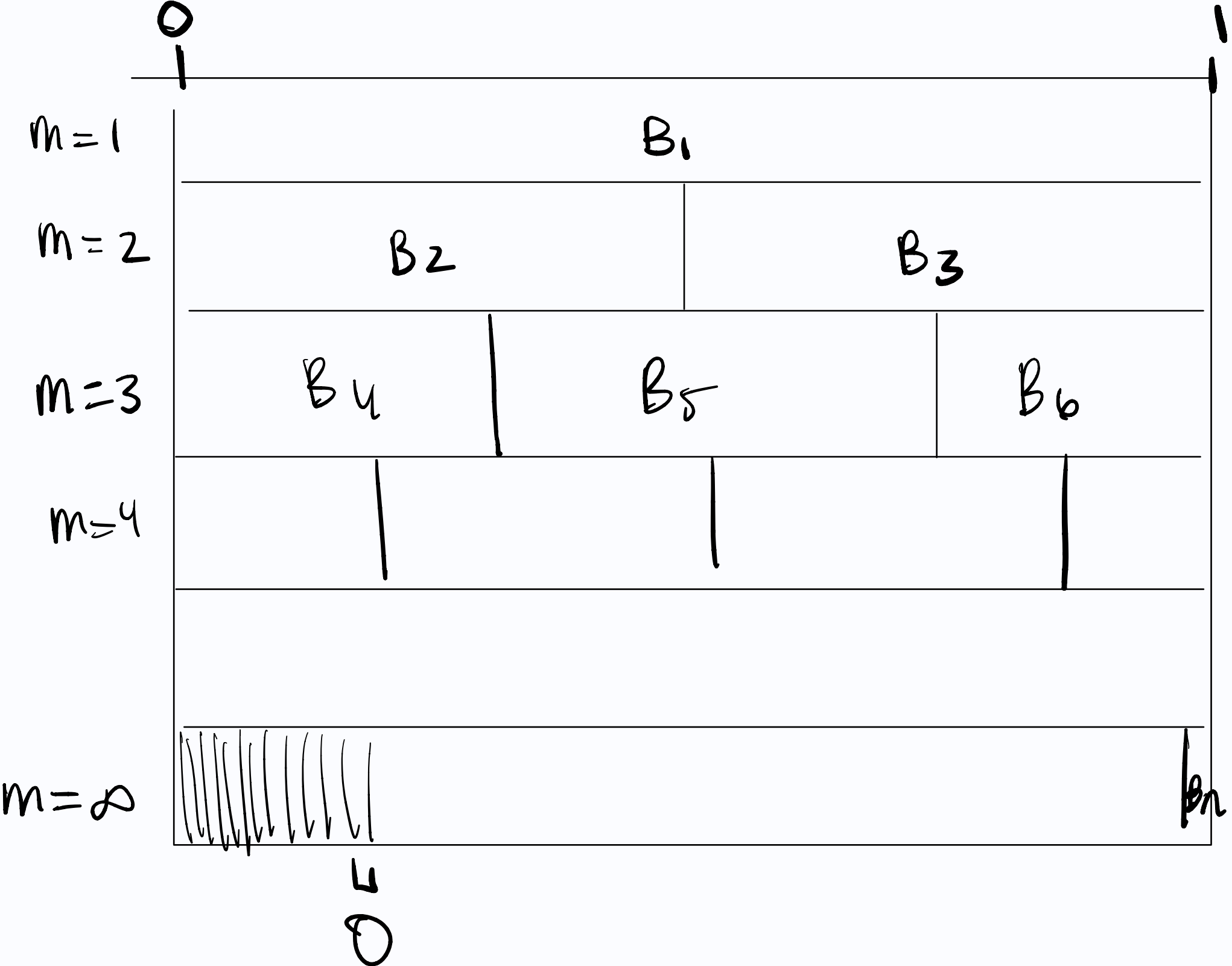
In general, row  $m$  contains  $m$  blocks each of length  $1/m$ , tiling  $[0, 1]$  completely. The blocks are indexed  $n = 1, 2, 3, \dots$  by reading left to right across rows. Set

$$X_n(\omega) = \mathbf{1}[\omega \in B_n].$$

Show that  $X_n \xrightarrow{P} 0$  but  $X_n \not\rightarrow 0$  almost surely.

# Convergence Almost Surely (with Probability One)

[0, 1]



# Convergence Almost Surely (with Probability One)

$$a) X_n \xrightarrow{P} 0 \quad X_n = \mathbb{I}(\omega \in B_n)$$

The  $n^{\text{th}}$  block  $B_n$  is in the  $m^{\text{th}}$  row where  $m \rightarrow \infty$  as  $n \rightarrow \infty$ .

As the number of blocks per row grows,  $P(X_n=1) = P(\omega \in B_n)$  shrinks.

$$P(|X_n - 0| \geq \epsilon) = P(X_n=1) = \frac{1}{m}$$

$$\lim_{n \rightarrow \infty} P(|X_n - 0| \geq \epsilon) = \lim_{m \rightarrow \infty} \frac{1}{m} = 0$$

↓  
as  $m \rightarrow \infty$  when  $n \rightarrow \infty$

$$\therefore X_n \xrightarrow{P} 0$$

# Convergence Almost Surely (with Probability One)

b)  $X_n \xrightarrow{a.s.} 0$  ?

For any  $\omega \in [0, 1]$ , for every row  $m$  there are  $1/m$  blocks and one of them must contain  $\omega$ .

$\Leftrightarrow X_n(\omega) = 1$  for one index  $n$  in every row  $m$ .

Consider:  $\omega = 0.7$

Row  $m_1$ :  $X_1(\omega) = 1$

$m_2$ :  $X_3(\omega) = 1$

$m_3$ :  $X_6(\omega) = 1$

$\vdots$

when  $m \rightarrow \infty$  (as  $n \rightarrow \infty$ ), there are infinitely many rows, and infinitely many  $X(\omega) = 1$ . This sequence of  $X_n(\omega)$ 's will never converge to 0.

$X_n \not\xrightarrow{a.s.} 0$

# Convergence Almost Surely (with Probability One)

# To Do

- Work on Assignment ~~5~~ due Wednesday June 17, 11:59pm on Gradescope.
- Read [Chapter 4.4](#) before next class.