Lecture 15

Convergence, Part I

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define convergence in probability
- Determine when a sequence converges in probability
- Define and apply the Weak Law of Large Numbers

1 Convergence in Probability

Convergence (Calculus)

- In calculus, you looked at the limit of a sequence of numbers, a_n, as n goes to infinity.

> 💡 **EXAMPLE**
>
> - $\lim_{n\to\infty} 1/n = 0$;
> - $\lim_{n\to\infty}(1 + 1/n)^n = e$;
> - $\lim_{n\to\infty}(1 + a/n)^n = e^a$ for all $a \in \mathbb{R}$;
> - $\lim_{n\to\infty}(1 + a_n/n)^n = e^a$ if $a_n \to a$.

- In probability, we look at the limit of a sequence of random variables, X_n, as n goes to infinity.
- This turns out to be more complicated, because there are different modes of convergence.

We will discuss 3 types of convergence.

Convergence in Probability

📖 **DEFINITION**

A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ converges in probability to a random variable X if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| < \epsilon) = 1.$$

- We can think of $a_n = \mathbb{P}(|X_n - X| < \epsilon)$ as a sequence of numbers that goes to one as n goes to infinity.
- This is the most similar to limits of sequences of numbers.
- Can also be written as $\mathbb{P}(|X_n - X| \geq \epsilon) \to 0$.
- Common notation: $X_n \overset{p}{\to} X$.

Convergence in Probability

💡 **EXAMPLE**

Suppose $\mathbb{P}(X_n = 1 - 1/n) = 1$ and $\mathbb{P}(Y = 1) = 1$. Show that the sequence $\{X_n\}$ converges in probability to Y.

Convergence in Probability

Convergence in Probability

💡 **EXAMPLE**

Let $U \sim \text{Unif}(0, 1)$ and define

$$X_n = U + B_n,$$

where $B_n \sim \text{Bern}(1/n)$ are independent Bernoulli random variables, also independent of U.

Show $X_n \xrightarrow{p} U$.

Convergence in Probability

Convergence in Probability

✏️ **EXERCISE: MAXIMUM OF IID UNIFORMS**

Let $U_1, U_2, \ldots$ be i.i.d. $\mathrm{Unif}(0, 1)$ random variables. Define $Y_n = \max\{U_1, \ldots, U_n\}$.

Show that $Y_n \xrightarrow{p} 1$.

Hint: $|Y_n - 1| > \epsilon$ if and only if $Y_n < 1 - \epsilon$.

Convergence in Probability

Weak Law of Large Numbers (WLLN)

THEOREM

Let $X_1, X_2, \ldots, X_n \ \ldots$ be independent and identically distributed (i.i.d) random variables with finite mean μ. Then,

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu.$$

Interpretation

The distribution of $\overline{X}_n$ gets more and more concentrated around μ as n increases.

Weak Law of Large Numbers (WLLN)

✏️ **EXERCISE: DICE**

Let X_n be the sum of the squares of n independent rolls of a fair six-sided die.

That is

$$X_n = \sum_{i=1}^{n} X_{n,i}^2,$$

where $X_{n,i}$ is the result of the i-th die roll.

Show that $n^{-1}X_n \xrightarrow{p} m$ for some m (find m explicitly).

Weak Law of Large Numbers (WLLN)

Proof of WLLN (with an extra condition)

📝 PROOF

- Assume that $\mathrm{Var}(X_i) = \sigma^2 < \infty$ (same for all i). This is not required for the WLLN, but it makes the proof easier.

Then, by Chebyshev's inequality, for all $\epsilon > 0$,

$$\mathbb{P}\left(\left|\overline{X}_n - \mu\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}(\overline{X}_n)}{\epsilon^2} = \frac{\sigma^2/n}{\epsilon^2} \to 0.$$

2 Convergence Almost Surely (with Probability One)

Convergence Almost Surely (with Probability One)

📖 DEFINITION

A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ converges **almost surely (or w.p. 1)** to a random variable X if for all $\epsilon > 0$,

$$\mathbb{P}\left(\lim_{n\to\infty} |X_n - X| < \epsilon\right) = 1.$$

- This is a stronger notion of convergence than convergence in probability.
- This is equivalent to saying that $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$.
- Writing this statement more explicitly, we are really demanding that

$$\mathbb{P}\left(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1.$$

- Common notation: $X_n \overset{a.s.}{\to} X$.

Convergence Almost Surely (with Probability One)

💡 EXAMPLE

Let $U \sim \text{Unif}(0,1)$ and

$$X_n = \begin{cases} 3 & U \leq \frac{2}{3} - \frac{1}{n} \\ 8 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} 3 & U \leq \frac{2}{3} \\ 8 & \text{otherwise} \end{cases}$$

Does $X_n \xrightarrow{a.s.} Y$?

Convergence Almost Surely (with Probability One)

Convergence Almost Surely (with Probability One)

💡 **EXAMPLE**

Let $Y \sim \text{Unif}(0, 1)$ and $X_n = Y^n$. Prove that $X_n \stackrel{a.s.}{\to} 0$.

Convergence Almost Surely (with Probability One)

Convergence Almost Surely (with Probability One)

💡 EXAMPLE

Let $U \sim \text{Uniform}(0, 1)$. Define the sequence $X_1, X_2, X_3, \ldots$ by partitioning $[0, 1]$ into successive blocks:

$$B_1 = [0, 1]; \quad B_2 = \left[0, \tfrac{1}{2}\right]; \quad B_3 = \left[\tfrac{1}{2}, 1\right]; \quad B_4 = \left[0, \tfrac{1}{3}\right]; \quad B_5 = \left[\tfrac{1}{3}, \tfrac{2}{3}\right]; \quad \ldots$$

In general, row m contains m blocks each of length $1/m$, tiling $[0, 1]$ completely. The blocks are indexed $n = 1, 2, 3, \ldots$ by reading left to right across rows. Set

$$X_n(\omega) = \mathbf{1}[\omega \in B_n].$$

Show that $X_n \xrightarrow{P} 0$ but $X_n \nrightarrow 0$ almost surely.

Convergence Almost Surely (with Probability One)

Convergence Almost Surely (with Probability One)

To Do

- Work on Assignment 5, due Wednesday June 17, 11:59pm on Gradescope.
- Read Chapter 4.4 before next class.