Lecture 16

Conditional Distributions and Independence

Grace Tompkins

Last modified — 05 Nov 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate and utilize conditional distributions
- Determine whether random variables are independent based on their conditional and joint distributions

1 Conditional Distributions

Conditioning on Discrete RVs

📖 **DEFINITION**

Let Y be a random variable, where X is a discrete random variable. Suppose that $\mathbb{P}(X = x) > 0$ for some x. Then the **conditional distribution** of Y given $X = x$ assigns to each set $A \subset \mathbb{R}$ the probability

$$\mathbb{P}(Y \in A \mid X = x) = \frac{\mathbb{P}(Y \in A, X = x)}{\mathbb{P}(X = x)}.$$

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

Here, we do not specify whether Y is continuous or discrete.

Conditioning on Discrete RVs

📖 **DEFINITION**

If X and Y are *both* discrete random variables, then the **conditional PMF** of Y given $X = x$ is defined by

$$p_{Y|X}(y \mid x) = \frac{\mathbb{P}(Y = y, X = x)}{\mathbb{P}(X = x)} = \frac{p_{X,Y}(x, y)}{p_X(x)}.$$

Note: we can rearrange this equation to get

$$p_{X,Y}(x, y) = p_{Y|X}(y \mid x) p_X(x)$$

Conditioning on Discrete RVs

- Let X and Y be the results of rolling two fair 6-sided dice.
- Let $V = X + Y$ and $W = \max\{X, Y\}$.

The joint PMF of W and V is

$W \setminus V$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1/36	0	0	0	0	0	0	0	0	0	0
2	0	2/36	1/36	0	0	0	0	0	0	0	0
3	0	0	2/36	2/36	1/36	0	0	0	0	0	0
4	0	0	0	2/36	2/36	2/36	1/36	0	0	0	0
5	0	0	0	0	2/36	2/36	2/36	2/36	1/36	0	0
6	0	0	0	0	0	2/36	2/36	2/36	2/36	2/36	1/36

To condition on W, we look at a particular row, while to condition on V, we look at a particular column. Then renormalize by the sum of the row/column.

Conditioning on Discrete RV

$W \setminus V$	2	3	4	5	6	7	8	9	10	11	12
1	1/36	0	0	0	0	0	0	0	0	0	0
2	0	2/36	1/36	0	0	0	0	0	0	0	0
3	0	0	2/36	2/36	1/36	0	0	0	0	0	0
4	0	0	0	2/36	2/36	2/36	1/36	0	0	0	0
5	0	0	0	0	2/36	2/36	2/36	2/36	1/36	0	0
6	0	0	0	0	0	2/36	2/36	2/36	2/36	2/36	1/36

V = 8

EXAMPLE

Given the above table, what is $\mathbb{P}(W = 5 \mid V = 8)$ = $\frac{P(W=5 \cap V=8)}{P(V=8)}$

$P(V=8) = 1/36 + 2/36 + 2/36 = 5/36$

$P(W=5, V=8) = 2/36$

$= \frac{2/36}{5/36} = 2/5$

Conditioning on Discrete RVs

Conditioning on Continuous RVs

📖 **DEFINITION**

If X and Y are jointly absolutely continuous random variables, then the **conditional density** of Y given $X = x$, is the function

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)},$$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

valid for any $y \in \mathbb{R}$, and all x such that $f_X(x) > 0$.

Note: we can rearrange this equation to get

$$f_{X,Y}(x,y) = f_{Y|X}(y \mid x) f_X(x)$$

Conditioning on Continuous RVs

📖 **DEFINITION**

Let X and Y be jointly absolutely continuous random variables with joint PDF $f_{X,Y}$. The **conditional distribution** of Y given $X = x$ assigns to each set $A \subset \mathbb{R}$ the probability

$$\mathbb{P}(a \leq Y \leq b \mid X = x) = \int_a^b f_{Y|X}(y \mid x)\, \mathrm{d}y,$$

valid for all x such that $f_X(x) > 0$.

Conditioning on Continuous RVs

$0 \le y \le x < \infty$

EXAMPLE

Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \frac{1}{x} e^{-x} I_{\{0 < y \le x\}}(x,y). \qquad \text{(joint)}$$

a. Find the marginal PDF of X. What distribution does X have?

b. Find the conditional PDF of Y given $X = x$. What distribution does $Y \mid X$ have?

a) $f_X(x) = \int f(x,y)\,dy$

$= \int_0^x \frac{1}{x} e^{-x}\,dy$

$= \frac{1}{x} e^{-x} \int_0^x dy$

$= \frac{1}{x} e^{-x} x$

$= e^{-x} I_{(0,\infty)}(x)$ → Exp(1) distribution!

Conditioning on Continuous RVs

b) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$= \frac{\frac{1}{x}e^{-x} I_{\{0<y\le x<\infty\}}(x,y)}{e^{-x} I_{(0,\infty)}(x)} = \frac{\frac{1}{x}e^{-x} I_{(0,x]}(y) I_{(0,\infty)}(x)}{e^{-x} I_{(0,\infty)}(x)}$$

$$= \frac{1}{x} I_{(0,x]}(y) \qquad Y \sim \text{Unif}(0, x)$$

Conditioning on Continuous RVs

Exp: Z: $f_Z(z) = \lambda e^{-\lambda z} I_{[0,\infty)}(z)$

> ✏️ **EXERCISE: TRY THIS ONE**
>
> Let X and Y be independent with $X \sim \text{Exp}(1)$ and $Y \sim \text{Exp}(2)$. Let $S = X + Y$. Find the conditional density $f_{X|S}(X = x|S = s)$

step 1: Find the joint density

Since X and Y are independent, their joint density is

$$f_{XY}(x,y) = (1)e^{-(1)x} I_{[0,\infty)}(x)\,(2)e^{-2y} I_{[0,\infty)}(y)$$
$$= e^{-x} 2e^{-2y} I_{[0,\infty)}(x)\, I_{[0,\infty)}(y)$$

$X: [0,\infty)$
$Y: [0,\infty)$
$\Rightarrow S = X+Y = [0,\infty)$

★ we can do this sub. because X and Y are independent and summed

Conditioning on Continuous RVs

As $X+Y=S \rightarrow X=X \quad Y=S-X$

$$\hookrightarrow \quad f_{XS}(x,s) = e^{-x} 2e^{-2(s-x)} I_{[0,s)}(x)\, I_{[0,\infty)}(x)$$

The support changes as having $Y \geq 0$ implies $s-x \geq 0 \Rightarrow$

$\Rightarrow -x \geq -s$

$x \leq s$

$\Rightarrow$ $s \geq x$ or $x \leq s$

We still have $0 \leq x < \infty$, so we can combine these into one.

$$\hookrightarrow \quad f_{XS}(x,s) = e^{-x} 2e^{-2(s-x)} I_{[0,s)}(x)$$

Step 2 Find marginal density of $S = X+Y$

$$f_S(s) = \int_0^s 2e^{-2(s-x)} e^{-x}\, dx = 2e^{-2s} \int_0^s e^x\, dx = 2e^{-2s} e^x \Big|_0^s$$

$$= 2e^{-2s}(e^s - 1)$$

Conditioning on Continuous RVs

Step 2 Find marginal density of $S = X + Y$

$$f_S(s) = \int_0^s 2e^{-2(s-x)} e^{-x}\, dx = 2e^{-2s} \int_0^s e^x\, dx = 2e^{-2s} e^x \Big|_0^s$$

$$- 2e^{-2s}(e^s - 1)$$

$$= 2(e^{-s} - e^{-2s})$$

Step 3:

$$f_{X|S}(X = x \mid S = s) = \frac{f_{XS}(x,s)}{f_S(s)} = \frac{2e^{(x-2s)}}{2(e^{-s} - e^{-2s})} I_{[0,s)}(x)$$

$$= \frac{e^{x-s}}{1 - e^{-s}} I_{[0,s)}(x)$$

after some factoring

Conditional Distributions and Independence

Recall that if X and Y are indpendent, we have: $\mathbb{P}(Y \in A \mid X = x) = P(Y \in A)$ This implies:

THEOREM

$$p_{Y|X}(y \mid x) = p_Y(y)$$

and

$$f_{Y|X}(y \mid x) = f_Y(y)$$

when X and Y are indpendent.

Heuristics Types of Distributions of Random Variables

- Suppose that X and Y have a joint distribution $\mathbb{P}(X \in A, Y \in B)$.
- We can find the joint CDF, PMF, or PDF of X and Y.
- The marginal distribution of X is the distribution of X when we "ignore" Y.
- The conditional distribution of X given $Y = y$ is the distribution of X when we "fix" Y to be y.
- The marginal comes from "summing out" or "integrating out" the other variable along the rows or columns of the table.
- The conditional comes from "dividing out" the other variable. Fix a row or a column, and then renormalize (divide out).
- When we say $Y \mid X = x$, we usually specify a formula. Meaning, we write down a function that gives the conditional PMF or PDF of Y given $X = x$ for all x such that $p_X(x) > 0$ or $f_X(x) > 0$. Renormalizing the row of a table gives the conditional PMF for a specific value of x.

Checkpoint 5

You may have:

- HANDWRITTEN Cheat Sheet
- Non-graphing, non-programmable calculator
- Writing materials

Phones and laptops put away, please! You can have 20 minutes.

Reminders:

- **No class Monday November 9**. No checkpoint on Friday, either!
- **Midterm 2 is on Friday November 20**