Lecture 16

Convergence, Part II

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Define convergence in distribution
- Determine when a sequence converges in distribution
- Define and apply the Central Limit Theorem (CLT)

1 Convergence in Distribution

Convergence in Distribution

📖 **DEFINITION**

A sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ with CDFs F_n converges in distribution to a random variable X with CDF F if, for all t at which F is continuous,

$$\lim_{n\to\infty} F_n(t) = F(t).$$

CDFs

- We will see that this is the "weakest" notion of convergence.
- But it gets used more frequently than the others.
- Common notation: $X_n \xrightarrow{d} X$.

📚 **PROPOSITION**

If there exists $s > 0$ such that for all $t \in (-s, s)$ $m_{X_n}(t) \to m_X(t)$, then $X_n \xrightarrow{d} X$.

Convergence in Distribution

💡 EXAMPLE

Let $U \sim \text{Unif}(0,1)$, and let $U_n \sim \text{Unif}(0,1)$ all independent. Define

$$X_n = U_n + B_n$$

where $B_n \sim \text{Bern}(1/n)$ are independent Bernoullis, also independent of $U, U_1, U_2, \ldots$.

Then $X_n \xrightarrow{d} U$ but X_n does NOT converge in probability to U.

We have that, for all t,

$$m_{X_n}(t) = m_{U_n}(t) m_{B_n}(t) = \frac{e^t - 1}{t}\left(1 - \frac{1}{n} + \frac{1}{n}e^t\right) \to \frac{e^t - 1}{t} = m_U(t).$$

Xn=Un+Bn and Un is ind of Bn ($U_n \perp B_n$)

MGF uniform

MGF for bernouilli

$\lim_{n\to\infty} m_X(t) = m_U(t)$

$X_n \xrightarrow{d} U$

Convergence in Distribution

> 💡 **EXAMPLE**
>
> ….continued….

However,

$$\begin{aligned}
\mathbb{P}(|X_n - U| > \epsilon) &= \mathbb{P}\left(|U_n + B_n - U| > \epsilon\right) \\
&= (1 - 1/n)\mathbb{P}\left(|U_n - U| > \epsilon\right) + (1/n)\mathbb{P}\left(|U_n + 1 - U| > \epsilon\right) \\
&= (1 - 1/n)(1 - \epsilon)^2 + (1/n)a \qquad \text{for some } a \in [0, 1] \\
&\to (1 - \epsilon)^2 \neq 0.
\end{aligned}$$

$P(B_n=1)$

$P(B_n=0)$

$\therefore \quad X_n \overset{P}{\not\to} U$

Convergence in Distribution

✏️ **EXERCISE: MGF CONVERGENCE**

Let $X_n \sim \mathcal{N}(0, 1 + 1/n)$ for all n, mutually independent. Show that $X_n \xrightarrow{d} Z \sim \mathcal{N}(0,1)$ by examining the moment generating functions.

Hint: Recall that the MGF of $\mathcal{N}(\mu, \sigma^2)$ is $m(t) = e^{\mu t + \sigma^2 t^2/2}$.

μ σ^2

$$m_{X_n}(t) = e^{(0)t + (1+1/n)t^2/2}$$

$$= e^{(1+1/n)t^2/2} \to e^{(1+0)t^2/2} \text{ as } n \to \infty$$

$$= e^{t^2/2}$$ which is the MGF of a N(0,1) RV.

$\therefore \lim_{n\to\infty} m_{X_n}(t) = m_Z(t)$ and $X_n \xrightarrow{d} Z$

Convergence in Distribution

Course Evaluation

Please take 10 minutes to fill out the course evaluation. This will:

- Help inform future course offerings (I'm teaching this again in Fall)
- Provide feedback on my own teaching on where to improve
- Help me stay employed in this economy :–)

I want you to fill this out regardless of how you feel about this course. Constructive feedback is welcome - rude comments about things I cannot change are **not** welcome. Keep it honest but professional - thank you!

Relationships Between Different Types of Convergence

THEOREM

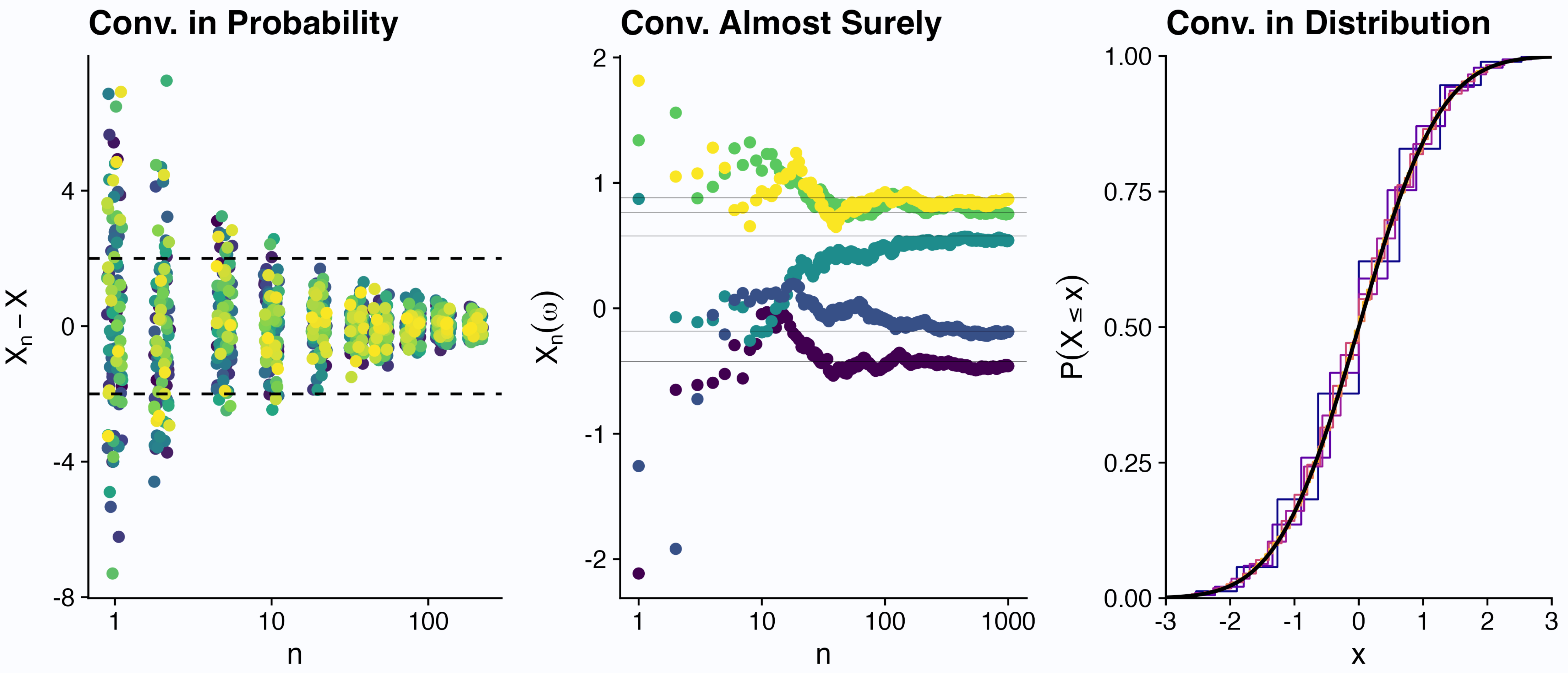
The following implications hold for any sequence of random variables $X_1, X_2, \ldots$ and any random variable X:

$$X_n \overset{a.s.}{\to} X \Longrightarrow X_n \overset{p}{\to} X \Longrightarrow X_n \overset{d}{\to} X.$$

Interpreting convergence

1. Convergence almost surely (with probability 1) is examining the sample path of $X_n(\omega)$. We need this **path** to converge to the value of $X(\omega)$ for almost all ω.
2. Convergence in probability involves the **joint distribution** of X_n and X: we are looking at the probability that $|X_n - X|$ is small. We hope this **probability** goes to one.
3. Convergence in distribution only involves the **marginal distribution** of X_n. We are looking at the **distribution** of X_n and hoping it gets closer and closer to the distribution of X as n increases.

Visualization of Convergence



- The probability that $X_n - X$ is large shrinks as n increases. (100 samples from $X_n - X$)
- For each ω, the sample path $X_n(\omega)$ gets closer and closer to $X(\omega)$ as n increases.
- The CDF of X_n gets closer and closer to the CDF of X as n increases.

Convergence of Maximum of I.I.D Uniforms

✏️ **EXERCISE: MAXIMUM OF IID UNIFORMS**

Let $U_1, U_2, \ldots$ be i.i.d. $\mathrm{Unif}(0,1)$ random variables. Define $Y_n = \max\{U_1, \ldots, U_n\}$.

Show that $n(1 - Y_n) \xrightarrow{d} \mathrm{Exp}(1)$.

Hints:

- Start by finding the CDF $F_n(t)$ of $n(1 - Y_n)$.
- Recall that the CDF of $\mathrm{Exp}(1)$ is $F(t) = 1 - e^{-t}$.

$$F_{n(1-Y_n)}(t) = P(n(1-Y_n) \le t)$$
$$= P(Y_n \ge 1 - t/n)$$
$$= 1 - P(Y_n < 1 - t/n)$$
$$= 1 - P(U_1 < 1 - t/n, U_2 < 1 - t/n, \ldots, U_n < 1 - t/n)$$

Convergence of Maximum of I.I.D Uniforms

$$F_{n(1-Y_n)}(t) = P(n(1-Y_n) \le t)$$
$$= P(Y_n \ge 1 - t/n)$$
$$= 1 - P(Y_n < 1 - t/n)$$
$$= 1 - P(U_1 < 1-t/n, U_2 < 1-t/n, \ldots, U_n < 1-t/n)$$
$$= 1 - \prod_{i=1}^{n} P(U_i < 1-t/n) \quad \text{since } U_i\text{'s are independent}$$
$$= 1 - \prod_{i=1}^{n} (1-t/n) \quad \text{since } P(U_i < 1-t/n) = 1-t/n \text{ by uniform dist.}$$
$$= 1 - (1-t/n)^n$$

$$\lim_{n\to\infty}\left(1-\left(1-\frac{t}{n}\right)^n\right) = 1 - e^{-t} \leftarrow \text{CDF of } \text{Exp}(1)!$$

$$\therefore \lim_{n\to\infty} F_{n(1-Y_n)}(t) = F_Z(t) \text{ where } Z \sim \text{Exp}(1)$$

$$\therefore n(1-Y_n) \xrightarrow{d} \text{Exp}(1)$$

Convergence in Probability vs Distribution

Discussion.

✏️ ~~EXERCISE: NORMAL CONVERGENCE, CONTINUED~~

Let $X_n \sim \mathcal{N}(0, 1 + 1/n)$, mutually independent, for all n. Does $X_n \xrightarrow{p} Z \sim \mathcal{N}(0,1)$? Justify your answer.

We showed earlier $X_n \xrightarrow{d} Z$. However, this does <u>not</u> converge in probability.

If we had said: $X_n = Z\sqrt{1 + 1/n}$, then you could show

$$P(|X_n - Z| > \epsilon) = 1 - (2\Phi(\epsilon/\sqrt{2 + 1/n}) - 1) > 0 \qquad \forall n,\ \epsilon > 0.$$

We could use MGF of the CDF to prove this.

2 Central Limit Theorem (CLT)

The Central Limit Theorem (CLT)

THEOREM

Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d random variables with finite mean μ and variance σ^2.

Then,

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

- People often say that $\overline{X}_n$ converges to a standard Gaussian.
- They mean that $\overline{X}_n$ **appropriately normalized** converges.

Interpretation

Probability statements about $\overline{X}_n$ can be approximated using a Normal distribution. It's the probability statements that we are approximating, not the random variable itself.

Equivalent Statements of the CLT

Define

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}.$$

- There are several forms of notation that all basically say the same thing.

$$\begin{aligned}
Z_n &\approx \mathcal{N}(0,1) \\
\overline{X}_n &\approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \\
\overline{X}_n - \mu &\approx \mathcal{N}\left(0, \frac{\sigma^2}{n}\right) \\
\sqrt{n}(\overline{X}_n - \mu) &\approx \mathcal{N}(0, \sigma^2) \\
\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} &\approx \mathcal{N}(0,1).
\end{aligned}$$

Equivalent Statements of the CLT

You should not say things like:

$$\overline{X}_n \xrightarrow{d} \mathcal{N}(\mu, \sigma^2/n).$$

- This grosses me out because you are taking a limit on the left-hand side where n goes to infinity, but the distribution on the right-hand side still depends on n.

Usefulness of the CLT

- In many situations, the exact distribution of $\overline{X}_n$, $\mathbb{P}(\overline{X}_n \leq x)$, is hard to determine exactly.
- The CLT allows us to approximate this value by

$$\mathbb{P}(\overline{X}_n \leq x) \approx \Phi\left(\frac{\sqrt{n}(x-\mu)}{\sigma}\right)$$

CDF of N(0,1)

with a respectable precision when n is large.

- Some people say that this approximation has acceptable precision when $n \geq 30$.
- Ignore those people.
- It would be more accurate to say "if $n < 30$, this approximation is probably bad".

Far Away Stars, Revisited

- Suppose that a radio telescope can measure the distance to a star.
- But due to atmospheric conditions, instrumental error, and movements of the earth, each measurement is a random variable with mean μ light years (the true distance) and variance 4 (square) light years.

σ^2

- An astronomer plans to take n independent measurements of the distance and use their average $\overline{X}_n$ as an estimate for the true distance.

✏️ **EXERCISE: FAR AWAY STARS (AGAIN!)**

Use the CLT to determine how many measurements the astronomer should make if they want the probability of a mismeasurement larger than 1 light year to be no more than 0.01?

Recall from Lecture 14: Chebyshev's said $n \geq 400$ measurements.

Recall $\Phi(z) = P(Z \leq z)$ where $Z \sim N(0,1)$

Far Away Stars, Revisited

$Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$

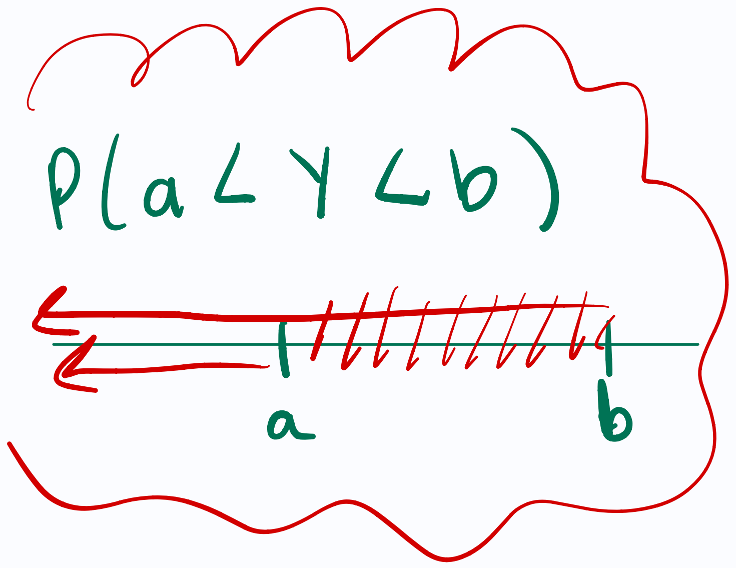
$\sigma^2 = 4$

$\sigma = 2$

$$P(|\bar{X}_n - \mu| < 1) = P(-1 < \bar{X}_n - \mu < 1)$$

$$= P\left(\frac{-1}{2/\sqrt{n}} < \frac{\bar{X}_n - \mu}{2/\sqrt{n}} < \frac{1}{2/\sqrt{n}}\right)$$

$$= P\left(\frac{\bar{X}_n - \mu}{2/\sqrt{n}} < \frac{1}{2/\sqrt{n}}\right) - P\left(\frac{\bar{X}_n - \mu}{2/\sqrt{n}} < \frac{-1}{2/\sqrt{n}}\right)$$



$$\approx \Phi\left(\frac{1}{2/\sqrt{n}}\right) - \Phi\left(\frac{-1}{2/\sqrt{n}}\right)$$

$$= \Phi\left(\frac{1}{2/\sqrt{n}}\right) - \left(1 - \Phi\left(\frac{1}{2/\sqrt{n}}\right)\right)$$

$$= 2\Phi\left(\frac{1}{2/\sqrt{n}}\right) - 1$$

$$\Rightarrow \quad 2\Phi\left(\frac{\sqrt{n}}{2}\right) - 1 = 0.99$$

$$2\Phi\left(\frac{\sqrt{n}}{2}\right) = 1.99$$

Far Away Stars, Revisited

$$\Phi\left(\frac{\sqrt{n}}{2}\right) - 1 = 0.99$$

$$2\Phi\left(\frac{\sqrt{n}}{2}\right) = 1.99$$

$$\Phi\left(\frac{\sqrt{n}}{2}\right) = 1.99/2$$

$$\frac{\sqrt{n}}{2} = \Phi^{-1}(0.995)$$

$$\sqrt{n} = \Phi^{-1}(0.995) \cdot 2$$

$$n = \left(\Phi^{-1}(0.995) \cdot 2\right)^2$$

$$= 26.54$$

↳ round up to 27.

Far Away Stars, Discussion

- Chebyshev's inequality suggests

$$n \geq 400$$

  independent observations.

- CLT suggests

$$n \geq 27$$

  independent observations

- Both are correct, but the CLT is more precise.
- To be fair, it used more information (the asymptotic distribution of the sample mean), which may or may not be accurate.
- Chebyshev's doesn't use any approximation, it's a guarantee.

Proof of the CLT

Let $Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$ as before.

Define $Y_i = (X_i - \mu)/\sigma$ for all i.

- Then, $Y_1, Y_2, \ldots$ are i.i.d. with mean 0 and variance 1, and we have $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i = Z_n$.
- By the proposition about MGFs, we need to show that $m_{Z_n}(t) \to m_Z(t)$ for all t, where $Z \sim \mathcal{N}(0, 1)$.

Suppose that Y_i has moment generating function $m_Y(t)$.

- Then, the moment generating function of $\sum Y_i$ is $m_Y(t)^n$.

Therefore, the moment generating function of Z_n is $m_Y(t/\sqrt{n})^n$.

Proof of the CLT, continued

Now, $m_Y'(0) = \mathbb{E}[Y_i] = 0$ and $m_Y''(0) = \mathbb{E}[Y_i^2] = 1$.

By Taylor's theorem, for all t,

$$
\begin{aligned}
m_Y(t) &= m_Y(0) + m_Y'(0)t + \frac{1}{2}m_Y''(0)t^2 + \cdots \\
&= 1 + 0 + \frac{t^2}{2} + \frac{t^3}{3!}m_Y'''(0) + \cdots \\
&= 1 + \frac{t^2}{2} + \frac{t^3}{3!}m_Y'''(0) + \cdots .
\end{aligned}
$$

$$
\implies \quad m_{Z_n}(t) = m_Y(t/\sqrt{n})^n = \left(1 + \frac{\frac{t^2}{2} + \frac{t^3}{3!n^{1/2}}m_Y'''(0) + \cdots}{n}\right)^n \to e^{t^2/2} = m_Z(t).
$$

[We used the fact that $\lim_{n\to\infty}(1 + a_n/n)^n = e^a$ when $a_n \to a$.]

Central Limit Theorem for I.I.D Sums

- The CLT states that when n is large, the distribution of

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \text{ is approximately } \mathcal{N}(0,1).$$

- This implies that when n is large, we can also say something about the distribution of $S_n = \sum_{i=1}^{n} X_i$.

$$\begin{aligned} 1 - \Phi(z) &= \lim_{n\to\infty} \mathbb{P}\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} > z\right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(\frac{(n\overline{X}_n - n\mu)}{n\sigma/\sqrt{n}} > z\right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} > z\right) \end{aligned}$$

Struggling Restaurants

- The daily sales on any given day of a restaurant is a random variable with mean of \$2500 and standard deviation of \$500.
- Assume that daily sales are independent random variables.

Let $T = \sum_{i=1}^{30} X_i$ be the total sales for 30 days

✏️ **EXERCISE: STRUGGLING RESTAURANTS**

Give an approximate value of the probability that the total sale for the 30 days will be over \$80,000.

Leave your answer in terms of Φ (the CDF of a standard Gaussian).

$$P(T > 80{,}000) = P\left(\frac{T - n\mu}{\sqrt{n}\sigma} > \frac{80{,}000 - n\mu}{\sqrt{n}\sigma}\right)$$

$$P(Z > z) = 1 - \Phi(z)$$

$$= P\left(\frac{T - (30)(2500)}{\sqrt{30}(500)} > \frac{80000 - (30)(2500)}{\sqrt{30}(500)}\right) \approx 1 - \Phi(1.8257)$$

$$\frac{T - (30)(2500)}{\sqrt{30}(500)} \approx N(0,1)$$

$$= 0.034$$

Exam Prep Advice

- Review class slides and create a first draft of your cheat sheet first.
- Review in class exercises, midterm, and assignments before attempting these problems.
- Try out these Exam Prep Problems unassisted before looking at the answer.
  - It is very easy to get a false sense of confidence if you look at the answer first! See how far you can get in the solution and allow yourself to get it wrong the first time.
  - If you're stuck, look for similar past questions and try to connect them.
  - Look at the solution after giving the problem a genuine attempt.
  - Consider adding/removing from your cheat sheet after trying these problems!
- Familiarize yourself with the distributions provided on the exam (see above), and your own cheat sheet.
- Ask for help if a solution is unclear. Piazza and office hours are resources that are here to help you.

Exam Rules

- The final exam is scheduled for **Monday June 22nd at 8:30am**. You can find the room location on Workday.
- It is 2 hours and 30 minutes and covers all content. There are **10 questions** of similar length and difficulty to the midterm.
- You may bring in **one** (1) “cheat sheet”:
  - Must be HAND WRITTEN with pen/pencil on said sheet of paper (not typed, not photo copied, not printed, not written on an iPad)
  - Must be on 8.5 by 11 inch sheet of paper or smaller d
  - You may write on both sides
  - **I will confiscate cheatsheets that do not follow these rules** 🥀
- Bring a non-programmable, non-graphing calculator.

The final page of your exam will also contain common distributions and general mean/variances (download the sheet here)

To Do

- Work on Assignment 4, due Wednesday June 17, 11:59pm on Gradescope.
- Next class: Review session! Let me know what you want to review by leaving a reply on the Piazza thread.