Lecture 17

Expectation

Grace Tompkins

Last modified — 13 Nov 2026

Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate expected values from continuous and discrete random variables/distributions
- Apply properties to functions of expected values

1 Expected Values

Expected Value of Random Variables

📖 **DEFINITION**

The **expected value** of a random variable $g(X)$ is defined by

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x & \text{if } X \text{ is absolutely continuous} \end{cases}$$

provided that the sum or integral exists.

- The sum in the discrete case is over all x such that $p_X(x) > 0$ (countable).
- The sum/integral exists when $\mathbb{E}[|g(X)|] < \infty$. Otherwise, we say that $\mathbb{E}[g(X)]$ does not exist.
- Note that you do not need to know the distribution/PMF/PDF/CDF of $g(X)$ to compute $\mathbb{E}[g(X)]$, only the distribution of X itself.

Expected Value of a Random Variable

> 💡 **EXAMPLE**
>
> Let X be the value of the face of a die when rolled. What is the expected value of X?

$P(X=1) = P(X=2) = \dots P(X=6) = 1/6$

$$E(X) = \sum_{i=1}^{6} x_i p(x_i)$$

$$= (1)(1/6) + (2)(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6)$$

$$= 3.5$$

→ this makes sense intuitively as 3.5 is halfway between 1 and 6 and we have equal probabilities for each event.

Expected Value of a Random Variable

Bin. PMF: $\binom{n}{x}\theta^x(1-\theta)^{n-x}\mathbb{I}_{[0,n]}(x)$

$R_X(x) =$

> 💡 **EXAMPLE**
>
> Let $X \sim \text{Binom}(n, \theta)$. What is $\mathbb{E}[X]$?
>
> *Hint: we can use kernel matching and use the fact that* $x\binom{n}{x} = n\binom{n-1}{x-1}$.

$$E(X) = \sum_{x=0}^{n} x \cdot p_X(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$= 0 + \sum_{x=1}^{n} x\binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$= \sum_{x=1}^{n} n\binom{n-1}{x-1}\theta^x(1-\theta)^{n-x} \quad \text{using hint}$$

$$= n\sum_{x=1}^{n}\binom{n-1}{x-1}\theta^x(1-\theta)^{n-x}$$

let $x-1=y$

$\Rightarrow x = y+1$

$$= n\sum_{y=0}^{n-1}\binom{n-1}{y}\theta^{y+1}(1-\theta)^{n-(y+1)}$$

$$= n\sum_{y=0}^{n-1}\binom{n-1}{y}\theta\theta^y(1-\theta)^{(n-1)-y}$$

$$= n\theta\sum_{y=0}^{n-1}\underbrace{\binom{n-1}{y}\theta^y(1-\theta)^{(n-1)-y}}_{\text{Bin}(n-1,\,\theta)}$$

$$= n\theta(1) = n\theta$$

Expected Value of a Random Variable

Expected Value of a Random Variable

✏️ **EXERCISE: GAMMA EXPECTATION**

Recall that the PDF of a RV $Y \sim \mathrm{Gam}(\theta, \beta)$ is given by

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

Suppose $X \sim \mathrm{Gamma}(\alpha, \lambda)$. What is $\mathbb{E}[X]$?

$$E(X) = \int x f_X(x)\, dx$$

$$= \int_0^\infty x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x}\, dx$$

Expected Value of a Random Variable

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} \, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^\infty \underbrace{\frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x} \, dx}$$

mult. by 1

PDF of $\text{Gam}(\alpha+1, \lambda)$, so integrates to 1

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$$

$$= \frac{\alpha \Gamma(\alpha)}{\Gamma(\alpha) \lambda}$$

$$= \boxed{\alpha/\lambda}$$

Expected Value of a Random Variable

✏️ **EXERCISE: MORE GAMMA EXPECTATIONS**

Let $X \sim \text{Gam}(\alpha, \lambda)$ where $\alpha > 0$ and $\lambda > 0$.

Let $t < \lambda$. Find $\mathbb{E}[\exp(tX)]$.

$$E(e^{tX}) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} \, dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} \int_0^\infty \underbrace{\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}}_{\text{Gam}(\alpha, \lambda-t)} \, dx$$

$$= \frac{\lambda^\alpha \Gamma(\alpha)}{\Gamma(\alpha)(\lambda-t)^\alpha} \quad (1)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^\alpha$$

or

$$\left(1 - \frac{t}{\lambda}\right)^{-\alpha}$$

Expected Value of a Random Variable

Important Properties

(Where the expected value exists.)

Linearity

for any $a, b, c \in \mathbb{R}$, any functions g and h, and any random variables X and Y.

$$\mathbb{E}[ag(X) + bh(Y) + c] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(Y)] + c$$

$E(X) = 4$

$E(2X+2) = 2(4) + 2$

Boundedness

If $a < g(x) < b$ for all x in the support of X, then $a < \mathbb{E}[g(X)] < b$.

Monotonicity

If $g(x) \leq h(x)$ for all x in the support of X, then $\mathbb{E}[g(X)] \leq \mathbb{E}[h(X)]$.

Independence

If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

NOTE: the converse is false.

→ you cannot verify independence using expected values!

Expected Value of A Function of Two Random Variables

✏️ **EXERCISE: EXPECATION OF TWO RV**

Let $X \sim U(0, \theta)$ and $Y \sim \text{Exp}(1)$ be independent.

Find $\mathbb{E}\left[\frac{1}{2}(X+Y)^2\right]$.

$$\mathbb{E}\left[\tfrac{1}{2}(X+Y)^2\right] = \tfrac{1}{2}\mathbb{E}\left[(X+Y)^2\right]$$

$$= \tfrac{1}{2}\mathbb{E}\left[X^2 + 2XY + Y^2\right]$$

$$= \tfrac{1}{2}\mathbb{E}(X^2) + \mathbb{E}(XY) + \tfrac{1}{2}\mathbb{E}(Y^2) \quad \text{by linearity}$$

$$= \tfrac{1}{2}\mathbb{E}(X^2) + \mathbb{E}(X)\mathbb{E}(Y) + \tfrac{1}{2}\mathbb{E}(Y^2) \quad \text{since } X, Y \text{ independent}$$

Expected Value of A Function of Two Random Variables

- $E(X^2) = \int_0^\theta x^2 \frac{1}{\theta-0}\,dx = \frac{1}{\theta}\int_0^\theta x^2\,dx = \frac{1}{\theta}\frac{x^3}{3}\Big|_{x=0}^{x=\theta} = \frac{1}{\theta}\frac{\theta^3}{3} = \frac{\theta^2}{3}$

- $E(X) = \int_0^\theta x\frac{1}{\theta}\,dx = \frac{\theta}{2}$

- $E(Y) = \int_0^\infty y e^{-y}\,dy = 1$

- $E(Y^2) = \int_0^\infty y^2 e^{-y}\,dy = \Gamma(3) = 2$

* make sure you can show this :)

$\therefore$

$$E\left(\tfrac{1}{2}(X+Y)^2\right) = = \tfrac{1}{2}E(X^2) + E(X)E(Y) + \tfrac{1}{2}E(Y^2)$$

$$\tfrac{1}{2}\left(\frac{\theta^2}{3}\right) + \frac{\theta}{2}(1) + \tfrac{1}{2}(2) = \frac{\theta^2}{6} + \frac{\theta}{2} + 1$$

Scalar-valued Functions of Multiple Random Variables

THEOREM

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function.

If X and Y are both discrete random variables, then

$$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y).$$

If X and Y are jointly absolutely continuous random variables, then

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) \mathrm{d}x \mathrm{d}y.$$

joint distributions

Product of Expectations

📐 **THEOREM**

- If X and Y are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$.

📝 **PROOF**

Suppose that X and Y are jointly absolutely continuous random variables with joint PDF $f_{X,Y}(x,y)$. Note that $\mathbb{E}[g(X)h(Y)]$ is a scalar-valued function of X and Y.

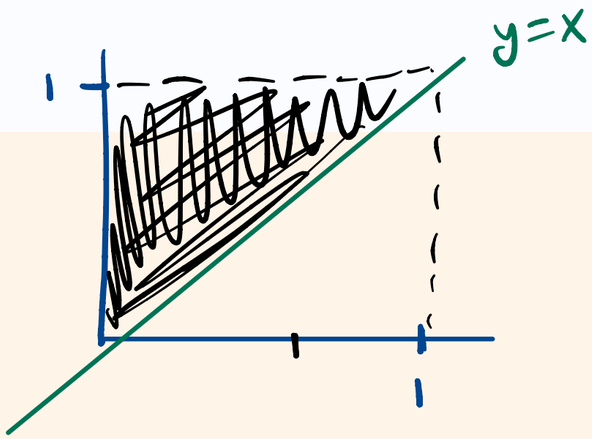
$$\mathbb{E}[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y) f_X(x) f_Y(y)\,dx\,dy \quad \text{since } X, Y \text{ indep.}$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\,dx \int_{-\infty}^{\infty} h(y) f(y)\,dy$$

$$= E(g(X)) \cdot E(h(Y))$$

Expectations from a Joint Distribution



✏️ **EXERCISE: EXPECTATION OF A CONTINUOUS JOINT DISTRIBUTION**

Let X and Y have joint PDF

$$f_{X,Y}(x,y) = 8xyI_{\{0<x<y<1\}}(x,y).$$

a. Calculate $\mathbb{E}[X]$. $= 8/15$

b. Calculate $\mathbb{E}[Y]$.

c. Calculate $\mathbb{E}[XY]$.

c)

$$E(XY) = \int_0^1 \int_0^y xy \; 8xy \; dx\,dy = 8 \int_0^1 \int_0^y x^2 y^2 \, dx\,dy$$

$$= 8 \int_0^1 \frac{x^3}{3} y^2 \, dy \Big|_{x=0}^{x=y}$$

Expectations from a Joint Distribution

$$E(XY) = \int_0^1 \int_0^y xy \; 8xy \; dx\,dy = 8 \int_0^1 \int_0^y x^2 y^2 \, dx\,dy$$

$$= 8 \int_0^1 \frac{x^3}{3} y^2 \, dy \Big|_{x=0}^{x=y}$$

$$= 8 \int_0^1 \frac{y^5}{3} \, dy$$

$$= \frac{8}{3} \left(\frac{y^6}{6} \right) \Big|_{y=0}^{y=1}$$

$$= \frac{8}{3} \left(\frac{1}{6} \right) = \boxed{\frac{4}{9}}$$

Expectations from a Joint Distribution

b) $E(Y) = \int_0^1 \int_0^y Y \, 8xy \, dx \, dy$

$= \int_0^1 \int_0^y 8xy^2 \, dx \, dy$

$= \int_0^1 4x^2y^2 \Big|_{x=0}^{x=y} dy$

$= \int_0^1 4y^4 \, dy$

$= \frac{4}{5}$

Expectations from a Joint Distribution

✏️ **EXERCISE: EXPECTATION OF A DISCRETE JOINT DISTRIBUTION**

Consider the following joint distribution:

$\mathbb{P}(X, Y)$	1	2	3
0	0.10	0.05	0.05
1	0.10	0.20	0.10
2	0.05	0.10	0.25

Find $\mathbb{E}[XY]$.

$$E(XY) = \sum_x \sum_y xy\, p_{xy}(x,y)$$

$$= (0)(1)(0.10) + (0)(2)(0.05) + (0)(3)(0.05)$$
$$+ (1)(1)(0.10) + (1)(2)(0.20) + (1)(3)(0.10)$$
$$+ (2)(1)(0.05) + (2)(2)(0.10) + (2)(3)(0.25) = 0 + 0.8 + 2 = 2.8$$

This is the end of the Midterm 2 content

- **Midterm 2 is in class on Friday November 20** (testing sections 2.4 (continuous RV/distributions) to 3.2 (today's class))
- Start prepping your cheat sheet
  - MUST BE HAND WRITTEN! No photo copies, typed, or tablet-written and printed sheets allowed
  - 8.5 inches x 11 inches or less
  - You can have one (1) single page, and can write on both sides
- Calculators will be inspected
  - Must be non-programmable, non-graphing, non-wifi-ing.