Lecture 17

Expectation

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Learning Outcomes

By the end of this lecture, students are anticipated to be able to:

- Calculate expected values from continuous and discrete random variables/distributions
- Apply properties to functions of expected values

1 Expected Values

Expected Value of Random Variables

📖 **DEFINITION**

The **expected value** of a random variable $g(X)$ is defined by

$$\mathbb{E}[g(X)] = \begin{cases} \sum_x g(x) p_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x & \text{if } X \text{ is absolutely continuous} \end{cases}$$

provided that the sum or integral exists.

- The sum in the discrete case is over all x such that $p_X(x) > 0$ (countable).
- The sum/integral exists when $\mathbb{E}[|g(X)|] < \infty$. Otherwise, we say that $\mathbb{E}[g(X)]$ does not exist.
- Note that you do not need to know the distribution/PMF/PDF/CDF of $g(X)$ to compute $\mathbb{E}[g(X)]$, only the distribution of X itself.

Expected Value of a Random Variable

> 💡 **EXAMPLE**
>
> Let X be the value of the face of a die when rolled. What is the expected value of X?

Expected Value of a Random Variable

💡 EXAMPLE

Let $X \sim \text{Binom}(n, \theta)$. What is $\mathbb{E}[X]$?

Hint: we can use kernel matching and use the fact that $x\binom{n}{x} = n\binom{n-1}{x-1}$.

Expected Value of a Random Variable

Expected Value of a Random Variable

✏️ **EXERCISE: GAMMA EXPECTATION**

Recall that the PDF of a RV $Y \sim \mathrm{Gam}(\theta, \beta)$ is given by

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x).$$

Suppose $X \sim \mathrm{Gamma}(\alpha, \lambda)$. What is $\mathbb{E}[X]$?

Expected Value of a Random Variable

Expected Value of a Random Variable

✏️ **EXERCISE: MORE GAMMA EXPECTATIONS**

Let $X \sim \text{Gam}(\alpha, \lambda)$ where $\alpha > 0$ and $\lambda > 0$.

Let $t < \lambda$. Find $\mathbb{E}[\exp(tX)]$.

Expected Value of a Random Variable

Important Properties

(Where the expected value exists.)

Linearity

for any $a, b, c \in \mathbb{R}$, any functions g and h, and any random variables X and Y.

$$\mathbb{E}[ag(X) + bh(Y) + c] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(Y)] + c$$

Boundedness

If $a < g(x) < b$ for all x in the support of X, then $a < \mathbb{E}[g(X)] < b$.

Monotonicity

If $g(x) \leq h(x)$ for all x in the support of X, then $\mathbb{E}[g(X)] \leq \mathbb{E}[h(X)]$.

Independence

If X and Y are independent, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

NOTE: the converse is false.

Expected Value of A Function of Two Random Variables

✏️ **EXERCISE: EXPECATION OF TWO RV**

Let $X \sim U(0, \theta)$ and $Y \sim \text{Exp}(1)$ be independent.

Find $\mathbb{E}\left[\frac{1}{2}(X+Y)^2\right]$.

Expected Value of A Function of Two Random Variables

Scalar-valued Functions of Multiple Random Variables

THEOREM

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function.

If X and Y are both discrete random variables, then

$$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y).$$

If X and Y are jointly absolutely continuous random variables, then

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) \mathrm{d}x \mathrm{d}y.$$

Product of Expectations

📐 **THEOREM**

- If X and Y are independent, then $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$.

📝 **PROOF**

Suppose that X and Y are jointly absolutely continuous random variables with joint PDF $f_{X,Y}(x, y)$. Note that $\mathbb{E}[g(X)h(Y)]$ is a scalar-valued function of X and Y.

Expectations from a Joint Distribution

✏️ **EXERCISE: EXPECTATION OF A CONTINUOUS JOINT DISTRIBUTION**

Let X and Y have joint PDF

$$f_{X,Y}(x,y) = 8xyI_{\{0<x<y<1\}}(x,y).$$

a. Calculate $\mathbb{E}[X]$.

b. Calculate $\mathbb{E}[Y]$.

c. Calculate $\mathbb{E}[XY]$.

Expectations from a Joint Distribution

Expectations from a Joint Distribution

Expectations from a Joint Distribution

✏️ **EXERCISE: EXPECTATION OF A DISCRETE JOINT DISTRIBUTION**

Consider the following joint distribution:

$\mathbb{P}(X, Y)$	1	2	3
0	0.10	0.05	0.05
1	0.10	0.20	0.10
2	0.05	0.10	0.25

Find $\mathbb{E}[XY]$.

This is the end of the Midterm 2 content

- **Midterm 2 is in class on Friday November 20** (testing sections 2.4 (continuous RV/distributions) to 3.2 (today's class))
- Start prepping your cheat sheet
  - MUST BE HAND WRITTEN! No photo copies, typed, or tablet-written and printed sheets allowed
  - 8.5 inches x 11 inches or less
  - You can have one (1) single page, and can write on both sides
- Calculators will be inspected
  - Must be non-programmable, non-graphing, non-wifi-ing.