STAT302

Review Session

Question 5c:

→ The integral does diverge and you should be able to justify why.

One option: Limit Comparison

as $x \to 0^+$, the function will behave like $\frac{1}{x}$

→ compare $\frac{e^{-x}}{x}$ to $\frac{1}{x}$

$$\lim_{x \to 0^+} \frac{e^{-x}/x}{1/x} = \lim_{x \to 0^+} e^{-x} = 1$$

Since this limit is finite and positive constant, both integrals will behave the same near 0.

$\therefore$ Since $\int_0^\infty \frac{1}{x}\,dx$ diverges, so does $\int_0^\infty \frac{e^{-x}}{x}\,dx$.

MGF of Poisson

hint: $\sum_{y=0}^{\infty} a^y/y! = e^a$ for some a

Recall PMF of a Poisson RV is:

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{when } X \sim Pois(\lambda)$$

for $x \in \{0,1,2,\dots\}$

Review of MGF:

$E(X) = \sum_x x\, p(x)$

$E(X^2) = \sum_x x^2 p(x)$

$E(e^{tx}) = \sum_x e^{tx} p(x)$

$$m_x(t) = E(e^{tx})$$

$$= \sum_x e^{tx} p_x(x)$$

$$= \sum_{x=0}^{\infty} \left(e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \right)$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \left(\frac{(\lambda e^t)^x}{x!} \right)$$

$$= e^{-\lambda} \left(e^{\lambda e^t} \right) \quad \text{by hint}$$

thus $m_x(t) = e^{\lambda(e^t - 1)}$

when $X \sim Poiss(\lambda)$

Textbook Problem 4.2.3

Let $W_1, W_2, \ldots$ be i.i.d with distribution Exp(3)
Prove for some n that

$$P(W_1 + W_2 + \ldots + W_n < n/2) > 0.999$$

→ technically you can use the CLT or WLLN to solve this.

We will use WLLN.

Let $S_n = W_1 + W_2 + \ldots + W_n$

$W_i \sim \text{Exp}(3)$, and independent.

↳ $E(W_i) = 1/\lambda = 1/3$

WLLN says: $\frac{S_n}{n} = \frac{\sum_{i=1}^{n} W_i}{n} \xrightarrow{P} E(W_i) = 1/3$

This implies

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \leq \epsilon\right) = 1$$

We are interested in:

$$P\left(S_n < \frac{n}{2}\right) = P\left(\frac{S_n}{n} < \frac{1}{2}\right)$$

Choose $\epsilon = 1/6$

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \leq 1/6\right) = 1$$

$\Rightarrow$

Choose $\epsilon = 1/6$

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \le 1/6\right) = 1$$

$$\left|\frac{S_n}{n} - \frac{1}{3}\right| \le 1/6 \Rightarrow -1/6 \le \frac{S_n}{n} - \frac{1}{3} \le 1/6$$

$$\Rightarrow \frac{S_n}{n} \le \frac{1}{2} \Rightarrow S_n \le \frac{n}{2}$$

You can argue:

$$P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \le 1/6\right) \le P\left(S_n < \frac{n}{2}\right)$$

$\lim \to 1$ as $n \to \infty$

$\Rightarrow \exists n$ such that

$$P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \le 1/6\right) > 0.999$$

$$\Rightarrow P\left(S_n < \frac{n}{2}\right) \ge P\left(\left|\frac{S_n}{n} - \frac{1}{3}\right| \le 1/6\right) > 0.99$$

Textbook 4.2.8

Let $Z_n \sim \text{Unif}(0, n)$

Let $W_n = \dfrac{5Z_n}{Z_n+1}$

$W = 5.$

Prove that $W_n \xrightarrow{P} W$

WTS: $\lim_{n\to\infty} P(|W_n - 5| > \epsilon) = 0$

Notice: $|W_n - 5| = \left| \dfrac{5Z_n}{Z_n+1} - 5 \right|$

$= \left| \dfrac{5Z_n - 5(Z_n+1)}{Z_n+1} \right|$

$= \left| \dfrac{-5}{Z_n+1} \right|$

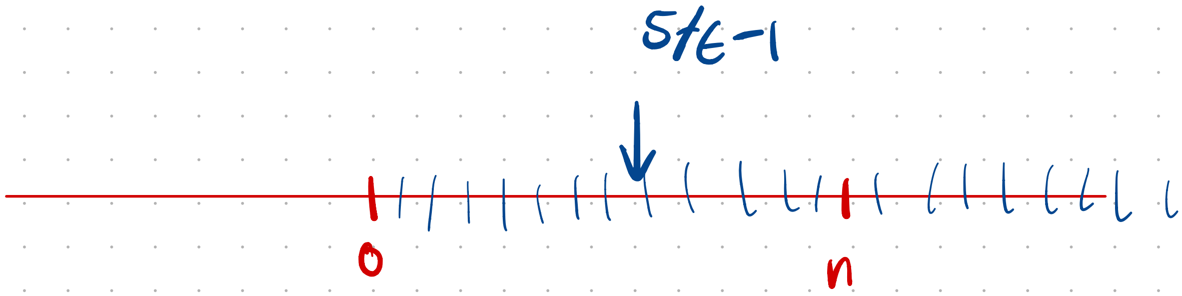
$= \dfrac{5}{Z_n+1}$

$\lim_{n\to\infty} P(|W_n - 5| > \epsilon) = \lim_{n\to\infty} P\left(\dfrac{5}{Z_n+1} > \epsilon\right)$

$= \lim_{n\to\infty} P(Z_n+1 < 5/\epsilon)$

$= \lim_{n\to\infty} P(Z_n < 5/\epsilon - 1)$

$$= \lim_{n\to\infty} P\left(Z_n < 5/\epsilon - 1\right)$$



if $5/\epsilon - 1 < 0$, this probability is 0

$$= \lim_{n\to\infty} \frac{\max\left(5/\epsilon - 1,\ 0\right)}{n}$$ since $Z_n \sim \text{Unif}(0, n)$

$$= 0$$

$\therefore\ W_n \xrightarrow{p} W$ $(=5)$

Textbook 4.3.2

$Y \sim \text{Unif}(0,1)$

$X_n = Y^n$

Prove $X_n \xrightarrow{a.s} 0$

$Y \in [0,1)$ $\lim_{n\to\infty} X_n = \lim_{n\to\infty} Y^n = 0$

$Y = 1$: $\lim_{n\to\infty} X_n = \lim_{n\to\infty} 1^n = 1$

But, $Y \sim \text{Unif}(0,1)$ which is continuous, so $P(Y=1) = 0$, so we don't need to worry about this case :)

$$X_n \xrightarrow{a.s.} 0$$

Textbook 4.4.4

Let W_n have density $\dfrac{1 + x/n}{1 + \frac{1}{2n}}$ for $0 < x < 1$, 0 o/w

where $W \sim \text{Unif}(0,1)$.

Show $\{W_n\} \xrightarrow{d} W$ hint: use the CDF

WTS. $F_{W_n}(x) \to F_W(x)$

For $x \leq 0$, $F_{W_n}(x) = 0$

$x \geq 1$ $F_{W_n}(x) = 1$

For: $0 < x < 1$

$$F_{W_n}(x) = P(W_n \leq x)$$

$$= \int_0^x \frac{1 + t/n}{1 + 1/2n} \, dt$$

$$= \frac{1}{1 + 1/2n} \int_0^x \left(1 + t/n\right) dt$$

$$= \frac{1}{1 + 1/(2n)} \left(x + x^2/2n\right)$$

$$= \frac{x + x^2/2n}{1 + 1/2n}$$

When $x \le 0$, $F_{W_n}(x) = F_W(x) = 0$

$x \ge 1$, $F_{W_n}(x) = F_W(x) = 1$

$0 < x < 1$:

$$\lim_{n\to\infty} F_{W_n}(x)$$

$$= \lim_{n\to\infty} \left[\frac{x + x^2/2n}{1 + 1/2n} \right]$$

$$= x = F_W(x)$$

[cumulative prob for $X \sim \text{Unif}(0,1)$ where $0 < x < 1$]

$$\therefore \lim_{n\to\infty} F_{W_n} = F_W$$

$$\therefore W_n \xrightarrow{d} W$$

Question 23 (Exam Prep)

M = Patient has marker

T = Positive test

$P(M) = 0.03 \Rightarrow P(M^c) = 1 - 0.03 = 0.97$

$P(\text{Correct Result}) = 0.92$

$P(T|M^c) = 4P(T^c|M)$

a) $P(T|M^c)$, $P(T^c|M)$

$P(\text{Correct Result}) = P(T \cap M) + P(T^c \cap M^c)$

$$0.92 = P(T|M)P(M) + P(T^c|M^c)P(M^c)$$

$$0.92 = (1 - P(T^c|M))0.03 + (1 - P(T|M^c))0.97$$

$$0.92 = (1 - P(T^c|M))0.03 + (1 - 4P(T^c|M))0.97$$

$$\Rightarrow P(T^c|M) = 0.02$$

$$\Rightarrow P(T|M^c) = 4(0.02) = 0.08$$

b) $P(T) = P(T|M)P(M) + P(T|M^c)P(M^c)$

$$= (1 - P(T^c|M))P(M) + P(T|M^c)P(M^c)$$

$$= (1 - 0.02)(0.03) + (0.08)(0.97)$$

$$= 0.107$$

→ law of tot probability.

c) $P(M|T) = \dfrac{P(T|M)P(M)}{P(T)}$ } Bayes

$$= \frac{(0.98)(0.03)}{0.107} = 0.2748$$

The prob . . . . . is 27.48%.

Question 22

$N = 20$

$K = 8$

$n = 6$

Hypergeometric dist.

$$P(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$P(X=3) = \frac{\binom{8}{3}\binom{20-8}{6-3}}{\binom{20}{6}} = \frac{56 \times 220}{38760} =$$

$$= 0.3178$$

b) $P(X \leq 1) = P(X=0) + P(X=1)$

Question 12

let y_i, $i = 1, \dots, \dots$ i.i.d such that

$E(y_i) = \beta x_i$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i} = c \qquad c > 0$$

$\beta \in \mathbb{R}$

let $$Z_n = n^{-1/2} \sum_{i=1}^{n} \left(\frac{y_i}{x_i} - \beta \right)$$

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a) $$E(Z_n) = E\left[n^{-1/2} \sum_{i=1}^{n} \left(\frac{y_i}{x_i} - \beta \right) \right]$$

recall: $E(aW + b) = aE(W) + b$ where $a, b \in \mathbb{R}$

$E(W + Z) = E(W) + E(Z)$

$$= n^{-1/2} \sum_{i=1}^{n} E\left[\frac{y_i}{x_i} - \beta \right]$$

$$= n^{-1/2} \sum_{i=1}^{n} \left[\frac{E(y_i)}{x_i} - \beta \right]$$

$$= n^{-1/2} \sum_{i=1}^{n} \left(\frac{\beta x_i}{x_i} - \beta \right)$$

$$= n^{-1/2} (0)$$

$$= 0$$

b) $Var(Y_i) = \sigma^2$

let $Z_n = n^{-1/2} \sum_{i=1}^{n} \left(\frac{Y_i}{x_i} - \beta \right)$

$$Var(Z_n) = Var\left(n^{-1/2} \sum_{i=1}^{n} \frac{Y_i}{x_i} - \beta \right)$$

Recall: $Var(aW + b) = a^2 Var(W)$

$$= (n^{-1/2})^2 \, Var\left(\sum_{i=1}^{n} \left(\frac{Y_i}{x_i} - \beta \right) \right)$$

Recall $Var(W + Z) = Var(W) + Var(Z) + 2Cov(W, Z)$ — 0 if $W \perp Z$

$$= n^{-1} \sum_{i=1}^{n} Var\left(\frac{Y_i}{x_i} - \beta \right)$$ since Y_i's are independent

$$= n^{-1} \sum_{i=1}^{n} \left(\frac{1}{x_i} \right)^2 Var(Y_i)$$

$$= n^{-1} \sum_{i=1}^{n} \left(\frac{1}{x_i^2} \right) \sigma^2$$

$$= \frac{\sigma^2}{n} \sum_{i=1}^{n} \frac{1}{x_i^2}$$