Statistics practice solutions

Stat 406, W1 2020

5 October 2020

1. Suppose $Y_1, \ldots, Y_n$ are iid Normal$(\mu,\ 1)$. Write down the *likelihood* of μ.

$$L(\mu; \mathbf{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - \mu)^2}{2}\right) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i - \mu)^2\right).$$

2. Suppose $Y_1, \ldots, Y_n$ are *independent* but not identically distributed Normal$(\mu_i,\ 1)$. Write down the *likelihood* of μ.

$$L(\boldsymbol{\mu}; \mathbf{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_i - \mu_i)^2}{2}\right) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}(y_i - \mu_i)^2\right).$$

3. Consider problem 1. What is the expected value of Y_{25}?

$$E[Y_{25}] = \mu$$

4. Consider problem 2. What is the expected value of Y_{25}?

$$E[Y_{25}] = \mu_{25}$$

5. Suppose $\widehat{Z}$ is an estimator of ϕ. What is the definition of the *bias* of $\widehat{Z}$?

$$\text{bias}(\widehat{Z}) = E\left[\widehat{Z} - \phi\right] = E\left[\widehat{Z}\right] - \phi$$

6. Consider problem 1. What is the *bias* of Y_{25} as an estimator of μ?

$$\text{bias}(Y_{25}) = E\left[Y_{25} - \mu\right] = E\left[Y_{25}\right] - \mu = \mu - \mu = 0$$

7. Consider problem 1. What is the *variance* of Y_{25}?

$$\text{Var}[Y_{25}] = 1$$

8. Consider problem 1. What is the *bias* of $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ as an estimator of μ?

$$\text{bias}(\overline{Y}) = E\left[\overline{Y} - \mu\right] = E\left[\overline{Y}\right] - \mu = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] - \mu = \mu - \mu = 0$$

9. Consider problem 1. What is the *variance* of $\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$?

$$\text{Var}[\overline{Y}] = \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}[Y_i] = \frac{1}{n^2}\sum_{i=1}^{n} 1 = n/n^2 = 1/n$$

10. Consider problem 1. Assume n is even. Is the *variance* of $\widetilde{Y}_n = \frac{1}{n/2}\sum_{i=1}^{n/2} Y_{2i}$ larger or smaller than that of $\overline{Y}_n$? (Could you find the variance if asked?)

$$\mathrm{Var}[\widetilde{Y}] = \mathrm{Var}\left[\frac{1}{n/2}\sum_{i=1}^{n/2} Y_{2i}\right] = \frac{1}{(n/2)^2}\sum_{i=1}^{n/2}\mathrm{Var}[Y_{2i}] = \frac{1}{(n/2)^2}\sum_{i=1}^{n/2} 1 = (n/2)/(n/2)^2 = 2/n$$

11. Consider the previous problem. What is the expected value of $\widetilde{Y}_n$?

$$E[\widetilde{Y}_n] = \frac{1}{n/2}\sum_{i=1}^{n/2} E[Y_{2i}] = \mu$$

12. How does the bias of $\widetilde{Y}_n$ compare to the bias of $\overline{Y}_n$?
It's the same.

13. What is the MSE of $\overline{Y}_n$ *as an estimator of* μ? That is

$$E\left[(\overline{Y}_n - \mu)^2\right] = 1/n$$

14. What is the MSE of $\overline{Y}_n$ *as an predictor of* Y_{n+1}? That is

$$E\left[(\overline{Y}_n - Y_{n+1})^2\right] = 1/n + 1$$

15. Suppose $Y_1, \ldots, Y_n$ are independent Normal with means $x_i^\top \beta = \sum_{j=1}^{p} x_{ij}\beta_j$ and variance σ^2.
   a. Ignoring the trivial cases ($\mathbf{x}_i = \mathbf{x}_j \forall i \neq j$ or $\beta = 0$) are the Y's identically distributed?
   No, they have different means
   b. What is $E[Y_{25}]$?

$$E[Y_{25}] = x_{25}^\top \beta.$$

   c. What is $E\left[\sum_{i=1}^{n} Y_i\right]$?

$$E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E[Y_i] = \sum_{i=1}^{n} x_i^\top \beta = \left(\sum_{i=1}^{n} x_i\right)^\top \beta$$

   d. Let $\hat{\beta} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}$. In this formula, what is random?
   Only $\mathbf{y}$ *is random*
   e. What is $E[\hat{\beta}]$?

$$E[\hat{\beta}] = E\left[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}\right] = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top E[\mathbf{y}] = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{X}\beta = \beta$$

   f. What is $\mathrm{Var}[\hat{\beta}]$?

$$\begin{aligned}\mathrm{Var}[\hat{\beta}] &= \mathrm{Var}\left[(\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}\right] = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathrm{Var}\left[\mathbf{y}\right]\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1} \\ &= (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top(\sigma^2\mathbf{I}_n)\mathbf{X}(\mathbf{X}^\top\mathbf{X})^{-1} = \sigma^2(\mathbf{X}^\top\mathbf{X})^{-1}\end{aligned}$$