STAT 460/560: Statistical Inference 1 | **Winter Term 1 (Fall) 2026**

Assignment 1

Due date: Sept. 17, 11:59 pm. | *Instructor: Daniel McDonald and Benjamin Bloem-Reddy*

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you **must** write your answers by yourself. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Use of AI tools to produce full or partial answers to these questions or to related questions is strictly prohibited. This includes tools like Co-pilot, auto-complete, etc. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see: the syllabus or contact the instructor.

1. Solutions may be handwritten or typeset with LaTeX (see the course website for a template). It must be legible. Difficult-to-read solutions will be marked down or ignored.
2. *Justify your answers formally. (Err on the side of too much detail so that the teaching team knows exactly what you're doing and why.)*
3. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
4. For questions that require computer simulations, you may use any language from the list of `(R, Julia, Python)`, with level of support provided by the teaching team decreasing further in the list. Please include in your PDF submission the relevant plots generated by your simulation code, and append your code (in PDF) to your submission. Please retain the code you used, as you may be asked for it if required.

Questions

1. A box contains w white balls and b black balls. A ball is chosen at random. The ball is then replaced, along with d more balls of the same color (as the chosen ball). Then another ball is drawn at random from the box. Show that that probability that the second ball is white does not depend on d.

   Solution:

$$\begin{aligned}\mathbb{P}(\text{2nd white}) &= \mathbb{P}(\text{2nd white} \mid \text{1st white})\,\mathbb{P}(\text{1st white}) + \mathbb{P}(\text{2nd white} \mid \text{1st black})\,\mathbb{P}(\text{1st black}) \\ &= \frac{w+d}{w+b+d}\frac{w}{w+b} + \frac{w}{w+b+d}\frac{b}{w+b} \\ &= \frac{(w+b+d)w}{(w+b+d)(w+b)} \\ &= \frac{w}{w+b},\end{aligned}$$

   which does not depend on d.

2. Similar to (Wasserman, 2004) 1.3.

   Show that

   (a) $\limsup_{n\to\infty} A_n = \{x \in S : \text{for all } n \in \mathbb{N} \text{ there exists } N \geq n \text{ such that } x \in A_N\}$.
   (b) $\liminf_{n\to\infty} A_n = \{x \in S : \text{there exists } N \in \mathbb{N} \text{ such that } x \in A_n \text{ for all } n \geq N\}$.

   Comments:

   - This is the reason why the set $\{\limsup_{n\to\infty} A_n\}$ is often denoted $\{x \in S : x \in A_n \text{ infinitely often}\}$ and $\{\liminf_{n\to\infty} A_n\}$ is often denoted $\{x \in S : x \in A_n \text{ eventually}\}$.
   - These sets are useful later on when studying convergence of estimators.

   Solution:

   (a)

$$\begin{aligned}
x \in \limsup A_n &\Leftrightarrow x \in \lim_{n\to\infty} \bigcap_{i=1}^{n} \bigcup_{k=i}^{\infty} A_k \\
&\Leftrightarrow x \in \left(\bigcup_{k=1}^{\infty} A_k\right) \cap \left(\bigcup_{k=2}^{\infty} A_k\right) \cap \cdots \\
&\Leftrightarrow \forall n,\ \exists N > n : x \in A_N \\
&\Leftrightarrow x \in A_n \text{ infinitely often.}
\end{aligned}$$

   (b)

$$\begin{aligned}
x \in \liminf A_n &\Leftrightarrow x \in \lim_{n\to\infty} \bigcup_{i=1}^{n} \bigcap_{k=i}^{\infty} A_k \\
&\Leftrightarrow x \in \left(\bigcap_{k=1}^{\infty} A_k\right) \cup \left(\bigcap_{k=2}^{\infty} A_k\right) \cup \cdots \\
&\Leftrightarrow \exists N : x \in A_N, A_{N+1}, \ldots \\
&\Leftrightarrow \exists N : x \in A_n\ \forall n > N \\
&\Leftrightarrow x \in A_n \text{ eventually.}
\end{aligned}$$

3. (a) Prove that if $A_1, A_2, \ldots$ are mutually exclusive, then $\mathbb{P}(A_n) \to 0$ as $n \to \infty$.
   (b) Suppose I flip a fair coin forever. Let A_n be the event that the n^{th} flip is a head. Since the coin is fair, $\mathbb{P}(A_n) = 1/2$. Notice that $\mathbb{P}(A_n) \not\to 0$ as $n \to \infty$. How, then, can the previous problem still be true? Each A_n refers to a different coin flip, after all. Explain your answer well.

   Solution:

   (a)

$$1 \geq \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = (\text{mutually exclusive}) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) \geq 0,$$

   Because each partial sum is non-negative and monotonically increasing, the series $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ converges to some value $c \in [0, 1]$ by the monotone convergence theorem. Noticing that $\mathbb{P}(A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i=1}^{n-1} \mathbb{P}(A_i)$, we have

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i) - \lim_{n\to\infty} \sum_{i=1}^{n-1} \mathbb{P}(A_i) = c - c = 0.$$

(b) The A_n's are independent, *not* mutually exclusive. In particular, if the first flip is heads, the second can still be heads.

4. Wasserman (2004) 2.7.

Solution: We have that

$$\mathbb{P}(Z > z) = \mathbb{P}(\min\{X, Y\} > z) = \mathbb{P}(X > z,\ Y > z) = \mathbb{P}(X > z)\mathbb{P}(Y > z),$$

where the last equality follows from independence. Since X and Y are both Uniform$(0, 1)$, we have $\mathbb{P}(X > z) = \mathbb{P}(Y > z) = 1 - z$ for $0 < z < 1$. Thus,

$$\mathbb{P}(Z > z) = (1 - z)^2.$$

The density of Z is then

$$f_Z(z) = \frac{d}{dz}\mathbb{P}(Z \le z) = \frac{d}{dz}(1 - \mathbb{P}(Z > z)) = \frac{d}{dz}(1 - (1 - z)^2) = 2(1 - z)I_{(0,1)}(z).$$

5. Prove that for a random variable Z, $P(Z = z) = 0$ if and only if the CDF of Z, $F_Z(z)$, is continuous at z.

Solution: F_Z is always continuous from the right ($\lim_{y\downarrow z} F_Z(y) = F_Z(z)$ by the definition of the CDF), so we need only deal with continuity from the left.

$$\begin{aligned}
\lim_{y\uparrow z} F_Z(y) &= \lim_{y\uparrow z} \mathbb{P}(Z \le z) = \lim_{y\uparrow z} \mathbb{P}(Z \le z) - \mathbb{P}(y < Z \le z) \\
&= \mathbb{P}(Z \le z) - \lim_{y\uparrow z} \mathbb{P}(y < Z \le z) = F_Z(z) - \mathbb{P}(Z = z) \\
&= F_Z(z)
\end{aligned}$$

if and only if $\mathbb{P}(Z = z) = 0$.

6. Let $Y = 3X + 2$ where X has pdf $f_X(x) = \frac{1}{\beta}e^{-x/\beta}$ (it has exponential distribution). Compute $\mathbb{E}[Y]$ four different ways:

(a) Use the pdf of Y and the definition of $\mathbb{E}[Y]$.

(b) Use the pdf of X and the definition of $\mathbb{E}[3X + 2]$.

(c) Use $\mathbb{E}[X]$ and the linearity of expectations.

(d) Use the moment generating function of Y.

Solution:

(a) First, we find the pdf of Y, using the transformation method. The transformation is $y = 3x + 2$, so $x = (y - 2)/3$ and $dx/dy = 1/3$. The support of Y is $[2, \infty)$ since $X \ge 0$. Therefore, the pdf of Y is given by:

$$f_Y(y) = \frac{1}{3}f_X\left(\frac{y-2}{3}\right) = \frac{1}{3\beta}\exp\left\{-\frac{y-2}{3\beta}\right\} I_{[2,\infty)}(y)$$

To find the expectation, we compute:

$$\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)dy = \int_2^{\infty} y\frac{1}{3\beta}\exp\left\{-\frac{y-2}{3\beta}\right\}dy \\
&= \int_0^{\infty} (z+2)\frac{1}{3\beta}\exp\left\{-\frac{z}{3\beta}\right\}dz \\
&= \left(\frac{\Gamma(2)(3\beta)^2}{3\beta} + 2\right)\int_0^{\infty} \text{Gamma}(2, 3\beta) \\
&= 3\beta + 2
\end{aligned}$$

(b)

$$\mathbb{E}[Y] = \mathbb{E}[3X+2] = \int_{-\infty}^{\infty} (3x+2)e^{-x/\beta} I_{[0,\infty)}(x)dx = 3\frac{\Gamma(2)\beta^2}{\beta}\int_0^{\infty} \text{Gamma}(2,\beta) + 2$$
$$= 3\beta + 2.$$

(c)
$$\mathbb{E}[Y] = \mathbb{E}[3X+2] = 3\mathbb{E}[X] + 2 = 3\beta + 2.$$

(d) The moment generating function of X is $M_X(t) = (1-\beta t)^{-1}$ for $t < 1/\beta$. The moment generating function of Y is then:

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(3X+2)}] = e^{2t}\mathbb{E}[e^{3tX}] = e^{2t}M_X(3t) = \frac{e^{2t}}{1-3\beta t}, \quad t < 1/(3\beta).$$

To find $\mathbb{E}[Y]$, we differentiate $M_Y(t)$ with respect to t and evaluate at $t = 0$:

$$M_Y'(t) = \frac{d}{dt}\left(\frac{e^{2t}}{1-3\beta t}\right) = \frac{2e^{2t}(1-3\beta t) + 3\beta e^{2t}}{(1-3\beta t)^2} = \frac{e^{2t}(2-3\beta t+3\beta)}{(1-3\beta t)^2}.$$

Evaluating at $t = 0$ gives:
$$M_Y'(0) = \frac{e^0(2+3\beta)}{(1-0)^2} = 2 + 3\beta.$$

7. Wasserman (2004) 3.4.

Solution: Let Y_i be the i^{th} jump of the particle, where $Y_i = -1$ with probability p and $Y_i = 1$ with probability $1-p$. Then we have that
$$X_n = \sum_{i=1}^n Y_i.$$

Since the jumps are independent, we have that

$$\mathbb{E}(X_n) = \mathbb{E}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \mathbb{E}(Y_i) = n\mathbb{E}(Y_1) = n(-p + (1-p)) = n(1-2p).$$

Similarly, since the jumps are independent, we have that

$$\text{Var}(X_n) = \text{Var}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \text{Var}(Y_i) = n\,\text{Var}(Y_1) = n(4p(1-p)).$$

References

Larry Wasserman. *All of Statistics*. Springer, New York, 2004.