

Assignment 3

Due date: Oct. 1, 11 p.m.

Instructor: Daniel McDonald and Benjamin Bloem-Reddy

Instructions

Academic integrity policy: I encourage you to discuss verbally with other students about the assignment. However, you **must** write your answers by yourself. If you discuss the assignment with anyone (a classmate or anyone else), you must say so at the top of your solutions. Use of AI tools to produce full or partial answers to these questions or to related questions is strictly prohibited. This includes tools like Co-pilot, auto-complete, etc. Also, refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see: the syllabus or contact the instructor.

1. Solutions may be handwritten or typeset with L^AT_EX (see the course website for a template). It must be legible. Difficult-to-read solutions will be marked down or ignored.
2. *Justify your answers formally. (Err on the side of too much detail so that the teaching team knows exactly what you're doing and why.)*
3. Submit your work by 11:59 pm (Vancouver time) on the due date to Canvas (.pdf).
4. For questions that require computer simulations, you may use any language from the list of (R, Julia, Python), with level of support provided by the teaching team decreasing further in the list. Please include in your PDF submission the relevant plots generated by your simulation code, and append your code (in PDF) to your submission. Please retain the code you used, as you may be asked for it if required.

Questions

1. Show that continuity is important for the Continuous Mapping Theorem. That is, construct a sequence X_n that converges in probability to a random variable X and a function ϕ is not continuous. Show that $\phi(X_n) \not\rightarrow \phi(X)$ in probability.

Solution: Let $X_1, X_2, \dots$ and X be such that $\mathbb{P}(X_n = 1/n) = 1$ for all n , and $\mathbb{P}(X = 0) = 1$. Define $\phi(x) = \mathbf{1}_{(0, \infty)}(x)$. Then

$$\mathbb{P}(|X_n - X| \geq \epsilon) = \mathbb{P}(1/n \geq \epsilon) = \mathbf{1}(n \leq 1/\epsilon) \xrightarrow{n \rightarrow \infty} 0$$

for any $\epsilon > 0$. So $X_n - X = o_{\mathbb{P}}(1)$.

But, for any $\epsilon \in (0, 1]$,

$$\mathbb{P}(|\phi(X_n) - \phi(X)| \geq \epsilon) = P(|1 - 0| \geq \epsilon) = 1$$

which does not go to 0. So $\phi(X_n) \not\rightarrow \phi(X)$ in probability. Note that ϕ is not continuous at 0.

2. [Wasserman \(2004\)](#) 5.13.

3. Let $X_1, X_2, \dots$ be i.i.d. $\text{Bern}(p)$ random variables and let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$.

- (a) Show that $\sqrt{n}(\bar{X}_n - p) \rightsquigarrow \mathbf{N}(0, p(1-p))$
- (b) Show that for $p \neq 1/2$, $\bar{X}_n(1 - \bar{X}_n)$ (an estimator of the variance) satisfies

$$\sqrt{n}(\bar{X}_n(1 - \bar{X}_n) - p(1-p)) \rightsquigarrow \mathbf{N}(0, (1-2p)^2 p(1-p)).$$

- (c) Suppose Y_n is a sequence of random variables that satisfies $\sqrt{n}(Y_n - \theta_0) \rightsquigarrow \mathbf{N}(0, \sigma^2)$ for some values θ_0 and σ^2 . Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi'(\theta_0) = 0$ but $\phi''(\theta_0)$ exists and is not equal to 0. In this setting, the **second-order Delta Method** gives

$$n(\phi(Y_n) - \phi(\theta_0)) \rightsquigarrow \frac{\sigma^2 \phi''(\theta_0)}{2} \chi_1^2.$$

where χ_1^2 is the χ -squared distribution with 1 degree of freedom.

Show that for $p = 1/2$, $n(\bar{X}_n(1 - \bar{X}_n) - 1/4) \rightsquigarrow \frac{1}{4} \chi_1^2$.

Solution:

- (a) We have $\mathbb{E}[X_i] = p$ and $\text{Var}(X_i) = p(1-p)$. By the Central Limit Theorem, we have

$$\sqrt{n}(\bar{X}_n - p) = \frac{\sum_{i=1}^n (X_i - p)}{\sqrt{n}} \rightsquigarrow \mathbf{N}(0, p(1-p)).$$

- (b) Let $\phi(x) = x(1-x)$. Then $\phi'(p) = 1-2p$ and $\phi''(p) = -2$. By the Delta Method, we have

$$\sqrt{n}(\phi(\bar{X}_n) - \phi(p)) = \phi'(\bar{X}_n) \sqrt{n}(\bar{X}_n - p) \rightsquigarrow \mathbf{N}(0, (1-2p)^2 p(1-p)).$$

- (c) For $p = 1/2$, we have $\phi'(1/2) = 0$ and $\phi''(1/2) = -2$. By the second-order Delta Method, we have

$$n(\phi(\bar{X}_n) - \phi(1/2)) \rightsquigarrow \frac{p(1-p)\phi''(1/2)}{2} \chi_1^2 = \frac{1/4 \cdot (-2)}{2} \chi_1^2 = \frac{1}{4} \chi_1^2.$$

4. Let $X_1, \dots, X_n$ be i.i.d. $\text{Unif}(0, \theta)$ random variables. Consider the estimator $\hat{\theta}_n = 2\bar{X}_n$ of θ .

- (a) Find the bias and variance of $\hat{\theta}_n$.
- (b) Find the mean squared error of $\hat{\theta}_n$.
- (c) Is $\hat{\theta}_n$ a consistent estimator of θ ? Justify your answer.
- (d) Find the asymptotic distribution of $\hat{\theta}_n$.

Solution:

- (a) We have $\mathbb{E}[\hat{\theta}_n] = 2\mathbb{E}[\bar{X}_n] = 2(\theta/2) = \theta$, so the bias is 0. We have

$$\text{Var}(\hat{\theta}_n) = 4 \text{Var}(\bar{X}_n) = 4(\theta^2/12n) = \theta^2/(3n).$$

- (b) The mean squared error is given by $\text{MSE}(\hat{\theta}_n) = \text{Var}(\hat{\theta}_n) + \text{Bias}^2(\hat{\theta}_n) = \theta^2/(3n) + 0^2 = \theta^2/(3n)$.

- (c) Yes. Since the MSE of $\hat{\theta}_n$ goes to 0 as $n \rightarrow \infty$, then $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$.

- (d) We have that $\mathbb{E}[X_i] = \theta/2$ and $\text{Var}(X_i) = \theta^2/12$. By the Central Limit Theorem, we have

$$\sqrt{n}(\bar{X}_n - \theta/2) \rightsquigarrow \mathbf{N}(0, \theta^2/12).$$

Therefore, we have

$$\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}(2\bar{X}_n - \theta) = 2\sqrt{n}(\bar{X}_n - \theta/2) \rightsquigarrow \mathbf{N}(0, \theta^2/3).$$

5. Let $X_1, \dots, X_n$ be i.i.d. $\text{Unif}(0, \theta)$ random variables. Consider the estimator $\hat{\theta}_n = \max\{X_1, \dots, X_n\}$ of θ .

- (a) Find the bias and variance of $\hat{\theta}_n$.
- (b) Find the mean squared error of $\hat{\theta}_n$.
- (c) Is $\hat{\theta}_n$ a consistent estimator of θ ? Justify your answer.
- (d) Find the asymptotic distribution of $\hat{\theta}_n$.

Solution: We have that $\hat{\theta}_n = \max\{X_1, \dots, X_n\}$. The cumulative distribution function of $\hat{\theta}_n$ is given by

$$F_{\hat{\theta}_n}(x) = \mathbb{P}(\hat{\theta}_n \leq x) = \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) = \mathbb{P}(X_1 \leq x)^n = (F_{X_1}(x))^n.$$

Since $X_i \sim \text{Unif}(0, \theta)$, we have $F_{X_1}(x) = x/\theta$ for $0 < x < \theta$. Therefore, we have

$$F_{\hat{\theta}_n}(x) = (x/\theta)^n \mathbf{1}_{(0, \theta)}(x) + \mathbf{1}_{[\theta, \infty)}(x).$$

The probability density function of $\hat{\theta}_n$ is then given by

$$f_{\hat{\theta}_n}(x) = \frac{n(x/\theta)^{n-1}}{\theta} \mathbf{1}_{(0, \theta)}(x).$$

- (a) The bias of $\hat{\theta}_n$ is given by

$$\begin{aligned} \text{Bias}(\hat{\theta}_n) &= \mathbb{E}[\hat{\theta}_n] - \theta = \int_0^\theta x f_{\hat{\theta}_n}(x) dx - \theta \\ &= \int_0^\theta x \frac{n(x/\theta)^{n-1}}{\theta} dx - \theta = \frac{n}{\theta^n} \int_0^\theta x^n dx - \theta = \frac{n}{n+1} \theta - \theta = -\frac{\theta}{n+1}. \end{aligned}$$

The variance of $\hat{\theta}_n$ is given by

$$\begin{aligned} \text{Var}(\hat{\theta}_n) &= \mathbb{E}[\hat{\theta}_n^2] - (\mathbb{E}[\hat{\theta}_n])^2 = \int_0^\theta x^2 f_{\hat{\theta}_n}(x) dx - \left(\frac{n}{n+1} \theta\right)^2 \\ &= \frac{n}{\theta^n} \int_0^\theta x^{n+1} dx - \left(\frac{n}{n+1} \theta\right)^2 = \frac{n}{n+2} \theta^2 - \left(\frac{n}{n+1} \theta\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}. \end{aligned}$$

- (b) The mean squared error of $\hat{\theta}_n$ is given by

$$\text{MSE}(\hat{\theta}_n) = \text{Var}(\hat{\theta}_n) + \text{Bias}^2(\hat{\theta}_n) = \frac{n\theta^2}{(n+1)^2(n+2)} + \left(-\frac{\theta}{n+1}\right)^2 = \frac{2\theta^2}{(n+1)(n+2)}.$$

- (c) Yes. Since the MSE of $\hat{\theta}_n$ goes to 0 as $n \rightarrow \infty$, then $\hat{\theta}_n \xrightarrow{\mathbb{P}} \theta$.
- (d) To find the asymptotic distribution of $\hat{\theta}_n$, we can use the fact that $\hat{\theta}_n$ converges in probability to θ . Therefore, we can consider the standardized version of $\hat{\theta}_n$:

$$Z_n = a_n(\theta - \hat{\theta}_n),$$

for some sequence a_n that goes to infinity. Note that we are considering $\theta - \hat{\theta}_n$ instead of $\hat{\theta}_n - \theta$ because the latter is always negative. The cumulative distribution function of Z_n for $z > 0$ is given by

$$F_{Z_n}(z) = 1 - F_{\hat{\theta}_n}(\theta - z/a_n) = 1 - ((\theta - z/a_n)/\theta)^n = 1 - \left(1 - \frac{z}{a_n\theta}\right)^n.$$

Here, we can choose $a_n = n$ for the sequence, recognizing the exponential limit. As $n \rightarrow \infty$, we have

$$F_{Z_n}(z) \rightarrow 1 - e^{-z/\theta}, \quad z > 0,$$

and $F_{Z_n}(z) = 0$ for $z < 0$. Therefore, the asymptotic distribution of $\hat{\theta}_n$ is

$$n(\theta - \hat{\theta}_n) \rightsquigarrow \text{Exp}(\theta).$$

Here, $\text{Exp}(\theta)$ denotes the exponential distribution with mean θ and rate $1/\theta$.

6. [Wasserman \(2004\)](#) 7.6.

Solution: Let $Y_i = \mathbf{1}(a < X_i \leq b)$ for $i = 1, \dots, n$. Since

$$\hat{\theta} = \hat{F}_n(b) - \hat{F}_n(a) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(X_i \leq b) - \frac{1}{n} \sum_{i=1}^n \mathbf{1}(X_i \leq a) = \frac{1}{n} \sum_{i=1}^n Y_i,$$

$\hat{\theta}$ is just the sample mean of i.i.d. Bernoulli random variables Y_i with success probability

$$\theta = \mathbb{P}(a < X \leq b) = F(b) - F(a).$$

Since $Y_i \sim \text{Bernoulli}(\theta)$,

$$\text{Var}(\hat{\theta}) = \frac{1}{n} \text{Var}(Y_i) = \frac{\theta(1-\theta)}{n}.$$

Plugging in $\hat{\theta}$ for the unknown θ gives the estimated standard error:

$$\widehat{\text{se}}(\hat{\theta}) = \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}.$$

By the CLT, $\hat{\theta}$ is asymptotically normal around θ with this variance, so an approximate $1-\alpha$ confidence interval is

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}},$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution.

References

Larry Wasserman. *All of Statistics*. Springer, New York, 2004.